COUPLING \( N \) SINGLE-PHOTON EMITTERS TO A SINGLE CAVITY MODE

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A system of two-level atoms in a cavity is discussed. First, the Jaynes-Cummings model is discussed of which the concepts are then ported to a system of multiple two-level atoms. Especially the case of two level atoms is studied in more detail. This model of two atoms is then considered in a leaking cavity to see how it behaves in an open system. This open system is simulated in different ways. Finally the coherent evolution of the cavity and emitted photon is discussed. This is done to examine if the distance of an observing detector to the cavity has an effect on the evolution of the cavity itself.
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CHAPTER 1

INTRODUCTION

As pioneered independently by Serge Haroche and David Wineland, optical cavities prove to be an environment in which atoms can be manipulated with great precision (1). Photons can carry information over large distances and two-level atoms can be used to store information and optical cavities are great means to let the two interact. For example, one of the most striking processes those atoms undergo in a cavity field is the so called Rabi-Cycle in which the two-level atom periodically absorbs and emits a photon in the cavity.

This thesis can be divided in two parts. The first part begins with chapter 2 by briefly going over the concepts of the most simple of such systems namely the famous Jaynes-Cummings model in section 2.1. These concepts are then ported to a system with multiple atoms as described in section 2.2. Mainly a system of two atoms in an optical cavity is discussed in detail and it is shown for example that a two-atom system too has an equivalent of the Rabi-oscillation . This chapter is concluded with some thoughts on the possibility of creating a pure N-photon photon state in a cavity with N atoms in section 2.5.

The second part starting from chapter 3 then moves over to an open cavity with a semi transparent mirror. The dynamics change into the dynamics open systems as the cavity field now interacts with the vacuum field outside the cavity. The damping caused by the now leaking cavity is modelled in various ways. After some discussion about a couple of properties of open systems, the theory from C. Gardiner & M. Collett (2)(3) is reviewed briefly in section 3.2 and section 3.3 to show how the behaviour of an open cavity can be modelled with a Langevin equation which is then used to create the more practical quantum stochastical differential equations (QSDE) which in turn can be used to derive a Master equation for the system with two atoms. After this, an equivalent
of the Master equation is discussed in section 3.4, namely the quantum jump operator method which is a Monte Carlo simulation. But of special interest in the following chapter is the photon emitted by the cavity. However both the Master equation and the Quantum-Jump Monte Carlo method are used mainly to describe the evolution of the system inside the cavity. Therefore, a method is devised that explicitly incorporates the photon density just outside the cavity in section 3.5. And it is shown that this method reproduces the same evolution as the Master equation and the quantum-jump method.

A final thing in this chapter to be discussed is an at first sight peculiar evolution when this system of two atoms in a cavity begins it's evolution with only a single excited atom. Entanglement by dissipation is observed in this case.

A final chapter chapter 4 then actually discusses the case wherein a detector is placed at a distance from the cavity. The photon and cavity then evolve coherently until this photon is detected. This is done to examine the effect of the distance of the detector on the evolution of system in the cavity. A similar simulation wherein a Mach-Zehnder interferometer is placed between the cavity and the detector was to be simulated, but due to time constraints this has not been done. This proposed simulation is discussed in a final paragraph.
CHAPTER 2

NON LEAKING CAVITY

This chapter will discuss the evolution of a system consisting of atoms interacting with an optical cavity. The chapter starts with the Jaynes-Cummings model which is the simplest of such a system. This is used to demonstrate its different aspects which will later be generalised for multiple atoms in what is called the Tavis-Cummings model. More specifically a system of two atoms will be discussed in depth. Then the chapter will conclude with some notes on how multiple atoms can be used in the production of N photon Fock states.

2.1 JAYNES-CUMMINGS MODEL

The simplest model that describes a two-level atom interacting with a resonant cavity is the "Jaynes-Cummings" model (4). This model consists of a single two-level atom inside an optical cavity. The cavity can be just two mirrors that are assumed to be perfect which reflect the photon in the cavity indefinitely. This system is described in fig. 2.1. An other possible cavity is a ring cavity which consists of mirrors that reflect the along a closed path.

The Hamiltonian of a two-level atom with the energy difference between the two levels being $\hbar \omega_0$ interacting with a near resonant cavity field with frequency $\omega_c$ is given by:

$$\hat{H}_{JC} = \hat{H}_{\text{atom}} + \hat{H}_{\text{field}} + \hat{H}_{\text{interaction}}$$

(2.1.1)

with
Figure 2.1: A simple schematic of an optical cavity with a single atom. The atom interacts with the photon inside the cavity as described by the Jaynes-Cummings model.

\[
\hat{H}_{\text{atom}} = \hbar \omega_a \hat{\sigma}^z \quad (2.1.2)
\]

\[
\hat{H}_{\text{field}} = \hbar \omega_c \left( \hat{a}^\dagger \hat{a} + \frac{1}{2} \right) \quad (2.1.3)
\]

\[
\hat{H}_{\text{int}} = \hbar g (\sigma^+ + \sigma^-)(\hat{a}^\dagger + \hat{a}) \quad (2.1.4)
\]

For convenience the zero point energy in Equation 2.1.3 is usually ignored.

And since a two level atom is the archetype of a spin-1/2 system, the operators \( \sigma^\pm = |e\rangle \langle g|, \sigma^+ = |e\rangle \langle g| \) and \( \sigma^- = |g\rangle \langle e| \) are equivalents of the well known Pauli Matrices.

The operators \( \hat{a} \) and \( \hat{a}^\dagger \) are the raising and lowering operators of the cavity Fock state.

The interaction Hamiltonian 2.1.4 describes the interaction of the dipole of the atom with the electromagnetic field in the cavity. The coefficient \( g \) then has the dimension of a frequency. This dipole approximation is only valid near resonance (5). The interaction Hamiltonian consists of two energy conserving terms \( \hat{a}^\dagger \sigma^- \) and \( a \sigma^+ \) which each describe respectively the emission and absorption of a cavity photon by the atom. And there are also two non-energy-conserving terms \( \hat{a}^\dagger \sigma^+ \) and \( a \sigma^- \). \( H_{\text{int}} \) can be seen as a perturbation of a Hamiltonian \( H_0 = H_{\text{atom}} + H_{\text{field}} \). When one considers this interaction Hamiltonian in the
Interaction picture, the non-energy conserving terms oscillate extremely fast. The rotating wave approximation then makes the reasonable assumption that the contribution of these non-energy-conserving terms averages out to zero in any realistic time scale. One can go over to the interaction picture:

\[ \mathcal{H}_{\text{int}} = e^{iH_0t}H_{\text{int}} e^{-iH_0t} \]  \hspace{1cm} (2.1.5)

In the end, the Interaction Hamiltonian in the interaction picture is then given by:

\[ \mathcal{H}_{\text{int}} = \hbar g \left( a^\dagger \sigma^- e^{i(\omega_a - \omega_c)t} + a \sigma^+ e^{i(\omega_a - \omega_c)t} \right) \]  \hspace{1cm} (2.1.6)

The frequency difference in the exponents denotes the detuning \( \Delta \) of the atom to the cavity \( \Delta = (\omega_a - \omega_c) \). It is possible to diagonalise the interaction Hamiltonian to get the eigenvectors or dressed states of the system. With each dressed state, an energy eigenvalue is associated. It can be seen that for large detuning, the dressed states approach the eigenvalues of \( H_0 \). But unlike them, the dressed states are not degenerate at the detuning \( \Delta = 0 \). The interaction energy splits up the energies between the two dressed states by \( 2g \) which is the Rabi frequency; this is plotted in 2.2 with \( |1\rangle \) and \( |2\rangle \) denoting the two dressed states. It can be seen that for large detunings, the two dressed states approach asymptotically the eigenvectors of \( H_0 \) which is logical since the interaction weakens with increasing detuning. Since the dressed states approach a different eigenvector of \( H_0 \) for different signs of \( \Delta \) it can be seen that the interaction rotates the dressed state from one eigenvector of \( H_0 \) into the other.

The equation of motion in the interaction picture can be given by:

\[ i\hbar \frac{\partial \Psi}{\partial t} = \mathcal{H}_{\text{int}} \Psi \]  \hspace{1cm} (2.1.7)

with \( \Psi \) a vector spanned by the eigenvectors of \( H_0 \). Since the atom can either be in an excited state \( e \) or the ground state \( g \) and there can be \( n \) photons in the cavity, these eigenvectors are written as \( |e,n\rangle \) or \( |g,n\rangle \). And therefore \( \Psi \) can be given by:

\[ \Psi = \sum_{n} \left[ C_{e,n}(t)|e,n\rangle + C_{g,n}(t)|g,n\rangle \right] \]  \hspace{1cm} (2.1.8)
Figure 2.2: The energies or eigenvalues of the dressed states of the Jaynes-Cummings Hamiltonian in function of the detuning $\Delta$. 
Figure 2.3: The probability of finding the atom in an excited state $e$ or the ground state $g$ in a resonant cavity

Entering this generic solution in the equation of motion, one can find the relations of the coefficients $C$.

\[
\begin{align*}
\dot{C}_{e,n} &= -ig \sqrt{n + 1} e^{i\Delta} C_{g,n+1} \\
\dot{C}_{g,n+1} &= -ig \sqrt{n + 1} e^{-i\Delta} C_{e,n}
\end{align*}
\] (2.1.9)

It can be seen that the state during its evolution stays in the Hilbert space defined by a certain amount of excitations. The excitation number is a number that represents the amount of energy in the cavity, being the number of excited atoms and photons at any given time.

A final thing to add to the system is a driving field that can interact with the atom in the cavity. This adds some terms to the Hamiltonian. First, there is the energy in the driving field itself $\hbar \omega_d b^\dagger b$. Secondly there is the interaction energy of this driving field with the atoms in the cavity $\hbar g_d (b^\dagger \sigma^- + b \sigma^+)$. However, the degrees of freedom of the driving field are of no particular interest, only its effect on the cavity as described in the interaction energy
Hamiltonian is of importance. The interaction can then simply be written as $\hbar g_d (\sigma^- + \sigma^+)$. Wherein the strength of the beam can just be incorporated in the coefficient $g_d$. It looks as if this interaction does not conserve energy, but one must remember that the energy is taken or given to the driving field. Therefore this term does not conserve the excitation number of the system. The Hamiltonian in the interaction picture can be in the same way as before written as:

$$\hat{\mathcal{H}}_{\text{int}} = \hbar g \left( a^\dagger \sigma^- e^{-i\Delta t} + a \sigma^+ e^{i\Delta t} \right) + \hbar g_d \left( \sigma^- e^{i(\omega_d - \omega_a)t} + a \sigma^+ e^{i(\omega_a - \omega_d)t} \right)$$

(2.1.10)

How the driving field can be used for population inversions will be discussed in section 2.3

### 2.2 The Tavis-Cummings model

This chapter describes the interaction of multiple atoms with a resonant optical cavity, especially the case of 2 atoms is considered. The theory of a gas of radiating two level atoms is described extensively with the Dicke Model by treating this gas as a single quantum system. M.Tavis and F.W.Cummings then examined such a gas of N two level atoms coupled to a single cavity mode in what is now referred to as the Tavis-Cummings model. Since states of many spin-$1/2$ atoms are considered, in many articles, the atoms and their collective states are treated as coupled angular momenta. However, here only small amounts of atoms are considered and the bare states of the atoms (excited or in the ground state) are of more interest than their total configuration. Therefore it is more intuitive to use the bare states described by the excitations of the separate atoms and the Fock state of photons in the cavity.

For two atoms, the same reasoning can be followed as in the case of one atom. Assuming again only one polarisation of light in the cavity.

$$\hat{\mathcal{H}} = \hbar \omega_c a^\dagger a + \hbar \omega_a \sigma^+_1 + \hbar \omega_a \sigma^+_2 + \hbar g a \left( \sigma^+_1 e^{i \vec{k} \cdot \vec{r}_1} + \sigma^+_2 e^{i \vec{k} \cdot \vec{r}_2} \right) + \hbar g a^\dagger \left( \sigma^-_1 e^{-i \vec{k} \cdot \vec{r}_1} + \sigma^-_2 e^{-i \vec{k} \cdot \vec{r}_2} \right)$$

(2.2.1)

with $\vec{k}$ the wave vector of the cavity mode and $\vec{r}_1$ and $\vec{r}_2$ the locations of the atoms. The locations of the atoms are of some importance to the system since they give the coefficients an extra phase, but those phases have no actual influence on the evolution of the populations of the states.
The Hamiltonian with two atoms is qualitatively equivalent with eq. (2.1.1), but each atom has its separate contribution to the Hamiltonian. Just like in the one atom case, it is more convenient to discuss the system in the interaction picture. The eigenvectors of $H_0$ (being the first three terms of the Hamiltonian) are in this case $|ee, n\rangle, |eg, n\rangle, |ge, n\rangle$ and $|gg, n\rangle$. Each atom can be in the excited state or the ground state while the cavity may contain an arbitrary number of photons $n$. The Hamiltonian in the interaction picture then is:

$$\hat{H}_{\text{int}} = \hbar g \left( a\sigma^+_1 e^{i(\Delta_1 t + \vec{k} \cdot \vec{r}_1)} + a\sigma^+_2 e^{i(\Delta_2 t + \vec{k} \cdot \vec{r}_2)} \right. $$

$$\left. + a^+\sigma^-_1 e^{-i(\Delta_1 t + \vec{k} \cdot \vec{r}_1)} + a^+\sigma^-_2 e^{-i(\Delta_2 t + \vec{k} \cdot \vec{r}_2)} \right)$$

(2.2.2)

Wherein the $\Delta_1$ and $\Delta_2$ denote the detuning for the two atoms. Although in most cases the atoms are assumed to be identical. The equations of motion in terms of the coefficients corresponding to the eigen vectors of $H_0$ can be retrieved in the same way as eq. (2.1.9):

$$\begin{align*}
\dot{C}_{ee,n} &= -ig \sqrt{n + 1} e^{i(\Delta_1 t + \vec{k} \cdot \vec{r}_1)} C_{ge,n+1} - ig \sqrt{n + 1} e^{i(\Delta_2 t + \vec{k} \cdot \vec{r}_2)} C_{gg,n+1} \\
\dot{C}_{ge,n+1} &= -ig \sqrt{n + 1} e^{-i(\Delta_2 t + \vec{k} \cdot \vec{r}_2)} C_{ee,n} - ig \sqrt{n + 2} e^{i(\Delta_1 t + \vec{k} \cdot \vec{r}_1)} C_{gg,n+2} \\
\dot{C}_{ge,n+1} &= -ig \sqrt{n + 1} e^{-i(\Delta_1 t + \vec{k} \cdot \vec{r}_1)} C_{ee,n} - ig \sqrt{n + 2} e^{i(\Delta_2 t + \vec{k} \cdot \vec{r}_2)} C_{gg,n+2} \\
\dot{C}_{gg,n+2} &= -ig \sqrt{n + 2} e^{-i(\Delta_1 t + \vec{k} \cdot \vec{r}_1)} C_{ge,n+1} - ig \sqrt{n + 2} e^{-i(\Delta_2 t + \vec{k} \cdot \vec{r}_2)} C_{ge,n+1}
\end{align*}$$

(2.2.3)

The equation of motion consists out of sets of 4 coupled differential equations. As can be seen, just like in the Jaynes-Cummings model, the excitation number is conserved. It is far more laborious to integrate out these equations directly. An other way is to derive the evolution operator.

$$\hat{U} = e^{i\hat{H}_{\text{int}}/\hbar}$$

(2.2.4)

In the case of $\Delta = 0$ this can be done with a lot of bookkeeping by a Tailor expansion of eq. (2.2.4) and separating the terms with even and uneven powers. Since this is a $4 \times 4$ matrix, 16 terms will have to be derived which is still very labour intensive. A far more elegant way is proposed in (9). Although the exact equations are not of much practical use since numerical integration is far more flexible, it is worth displaying them to examine their structure.
In the case of 2 excitations, the matrix elements of eq. (2.2.4) can be given by:

\[
U_{11} = \frac{2 + \cos(tg\sqrt{6})}{3} \\
U_{12} = U_{13} = -\frac{i \sin(tg\sqrt{6})}{\sqrt{6}}a \\
U_{14} = \frac{1 + \cos(tg\sqrt{6})}{3}a^2 \\
U_{21} = U_{31} = -\frac{i \sin(tg\sqrt{6})}{\sqrt{6}}a^\dagger \\
U_{23} = U_{32} = -\frac{1 + \cos(tg\sqrt{6})}{2} \\
U_{22} = U_{33} = \frac{1 + \cos(tg\sqrt{6})}{2} \\
U_{41} = \frac{1 + \cos(tg\sqrt{6})}{3}(a^\dagger)^2 \\
U_{42} = U_{43} = -\frac{i \sin(tg\sqrt{6})}{\sqrt{6}}a^\dagger \\
U_{44} = \frac{1 + 2 \cos(tg\sqrt{6})}{3}
\]

(2.2.5) \hspace{1cm} (2.2.6) \hspace{1cm} (2.2.7) \hspace{1cm} (2.2.8) \hspace{1cm} (2.2.9) \hspace{1cm} (2.2.10) \hspace{1cm} (2.2.11) \hspace{1cm} (2.2.12) \hspace{1cm} (2.2.13)

As in the Jaynes-Cummings model, the system oscillates with a certain 2-atom Rabi frequency. This Rabi frequency in this system, if the detuning \( \Delta = 0 \) is \( g\sqrt{2(2N - 1)} \) with \( N \) the excitation number.

Just like the Jaynes-Cummings model, one can diagonalise the complete interaction Hamiltonian to get the dressed states and their accompanying eigenvalues. For 2 excitations in the system, one has a 4 x 4 matrix. The eigenvalues are therefore the solutions of a 4th grade equation.

A more flexible way of writing eq. (2.2.2) is in a matrix notation. The \( \hat{\sigma}_-, \hat{\sigma}_+ \) and \( \hat{\sigma}_z \) operators are the Pauli matrices and in the case of multiple atoms, the operators \( \hat{\sigma}_-, \hat{\sigma}_+ \) and \( \hat{\sigma}_z \) are the same but should only act upon the i’th atom. The most convenient way of defining such operators in the case of \( n_a \) atoms are:

\[
\hat{\sigma}_i^+ = 1_2^{[i-1]} \otimes \hat{\sigma}_+ \otimes 1_2^{[n_a-i]} 
\]

(2.2.14)

with the exponent between square brackets denoting a Kronecker power (10).
Figure 2.4: The populations of the different states of two atoms in an optical cavity during 1 period.
Analogous for $\hat{\sigma}^-$ and $\hat{\sigma}^+$. The overall Hamiltonian can again be split up in as $H_0 + H_{int}$.

$$\hat{H}_0 = n\hbar\omega_c + \sum_i \hbar \frac{\hat{\sigma}_i^z}{2}\omega_c$$

$$\hat{H}_{int} = \sum_i \hbar \frac{\hat{\sigma}_i^z}{2}(\Delta_i) + \sum_i \hbar g_i \left( e^{i\Delta_i t} \hat{\sigma}_i^+ \hat{\alpha} + e^{-i\Delta_i t} \hat{\sigma}_i^- \hat{\alpha}^\dagger \right)$$

In the most practical case, both atoms are of the same kind and have therefore the same detuning $\Delta$ with respect to the cavity mode. The eq. (2.2.16) can be written as:

$$\hat{\mathcal{H}}_{int} = \begin{pmatrix} \Delta & \sqrt{n+1}g & \sqrt{n+1}g & 0 \\ \sqrt{n+1}g & 0 & 0 & \sqrt{n+2}g \\ \sqrt{n+1}g & 0 & 0 & \sqrt{n+2}g \\ 0 & \sqrt{n+2}g & \sqrt{n+2}g & -\Delta \end{pmatrix}$$

This matrix is spanned by the four eigenvectors of $H_0$: $|ee,n\rangle, |eg,n+1\rangle, |ge,n+1\rangle$ and $|gg,n+2\rangle$ in this order.

It is most interesting to see how the eigenvalues and their accompanying dressed states evolve in function of the detuning $\Delta$. This will be done in the most basic case happens when $n = 0$, when there are only 2 excitations in the system. E.g there are 2, 1 or 0 excited atoms with respectively 0, 1 or 2 photons in the cavity.

The characteristic polynomial of eq. (2.2.17) in the case of 2 excitations is given by:

$$\lambda^4 - (\Delta^2 + 6g^2)\lambda^2 + 2g^2\Delta\lambda = 0$$

which has a solution $\lambda = 0$ After dividing out $\lambda$ there are 3 other solutions left for the remaining third grade equation. Let:

$$\begin{cases} S = \sqrt[3]{\sqrt{\frac{(-\Delta^2 - 6)^3}{27} + \Delta^2} - \Delta} \\ T = i\sqrt[3]{-\sqrt{\frac{(-\Delta^2 - 6)^3}{27} + \Delta^2} - \Delta} \end{cases}$$

Then the solutions of the characteristic polynomial eq. (2.2.18) can be given by:
Figure 2.5: The energies of the dressed states of eq. (2.2.17) in function of the detuning $\Delta$.

\[
\begin{align*}
\lambda_1 &= S + T \\
\lambda_2 &= -\frac{1}{2}(S + T) + \frac{i}{2}\sqrt{3}(S - T) \\
\lambda_3 &= -\frac{1}{2}(S + T) - \frac{i}{2}\sqrt{3}(S - T) \\
\lambda_3 &= 0
\end{align*}
\]  

(2.2.20)

As can be seen in fig. 2.5, at $\Delta = 0$, two eigenvalues are degenerate at the origin, and the two non-zero eigenvalues have values $\pm \sqrt{6}g$. Just like in the single atom case, the interaction with a cavity splits up the energies corresponding to the dressed states. These energies are again the origin of the Rabi frequencies. A remarkable feature in fig. 2.5 is the fact that the dressed states are not symmetric about the y-axis. As $\Delta$ moves away from 0, the degeneracy of the two middle values are lifted, but with a different sign depending on the sign of $\Delta$. Secondly, the largest eigenvalue does not reach its minimum where the smallest eigenvalue reaches its maximum. They do both separately at $\Delta = 0.141$ and $\Delta = -0.141$ at the values $(2 + 0.141)$ and $-(2 + 0.141)$. The middle non-zero eigenvalue achieves its maximum and minimum at the
Figure 2.6: The evolution of the eigenvectors in the base spanned by the eigenvectors of $H_0$ in function of the detuning $D$.

points $\Delta = (2 + 0.141)$ and $\Delta = -(2 + 0.141)$ at energies 0.141 and 0.141. In units of $g$

This remarkable behaviour is to some extent understandable if one considers the behaviour of the dressed states in terms of the eigenvectors of $H_0$. The analytic expressions of these eigenvectors are complex and bulky and are therefore omitted. Instead their coefficients corresponding to $|ee, 0\rangle, |eg, 1\rangle, |ge, 1\rangle$ and $|gg, 2\rangle$ are plotted in fig. 2.6. It can be seen that the minimum and maximum of the two outer eigenvalues correspond to the crossing point of all coefficients. At this special point, all four coefficients attain the same value 0.5. Similarly, at the points where the middle non-zero eigenvalue reaches its maximum and minimum, three of the four coefficients intersect.

An other interesting fact can be seen with the dressed states $|1\rangle$ and $|4\rangle$. If one looks in fig. 2.5 one can see that just like in the one atom case, their eigenvalue approaches asymptotically towards the energy of a different eigenvector of $H_0$ for different signs of $\Delta$. This can be clearly seen in fig. 2.6. in this figure are only 3 coefficients visible per eigenstate. This is because two of them always assume the same value.
2.3 Driving the Atoms

A driving field can be added to the system. This driving field is generally a controllable field that can for example be used to excite or de-excite one or more atoms. The interaction of the atoms with this driving field is analogous to the interaction of the atoms with the cavity field. This adds a couple of terms to the Hamiltonian eq. (2.2.15) and eq. (2.2.16). The energy of the driving field can be given by \( \hbar \omega_d b^\dagger b \) and the coupling of the driving field to the atoms \( \hbar g_d (b^\dagger \sigma^- + b \sigma^+) \) with \( b^\dagger \) and \( b \) the creation and annihilation operators of the driving field.

It is assumed that the number of photons in the beam is sufficiently large that the absorption and emission of a photon from and to the beam does not noticeably change the intensity of the beam. Therefore \( b \approx b^\dagger \approx \sqrt{n} \). The constant factor \( \sqrt{n} \) can be absorbed into the coupling \( g_d \).

The interaction Hamiltonian in the interaction picture can in the same way be obtained as before:

\[
\hat{H}_{\text{int}} = \sum_i \hbar g_i \left( e^{i \Delta_i} \hat{\sigma}_i^+ a + e^{-i \Delta_i} \hat{\sigma}_i^+ a^\dagger \right) + \sum_i \hbar g_d \left( e^{i \Delta_d} \hat{\sigma}_i^+ + e^{-i \Delta_d} \hat{\sigma}_i^+ \right) \tag{2.3.1}
\]

This Hamiltonian does not conserve energy or the excitation number. This is because the driving field can excite or de-excite atoms without affecting the number of photons in the cavity.

The atoms in the cavity will under influence of this field oscillate with a Rabi frequency depending on the strength of this field. The driving field is assumed multiple orders of magnitude larger than the cavity field, so the frequency will be much higher than the Cavity Rabi Frequency.

Since the driving field adds or subtracts energy from the system, the excitation number isn’t a constant anymore. The main effect of this is that the equations of motion are now an infinite set of coupled differential equations. The equations have the same form as (2.2.3) but with extra terms according to the Hamiltonian (2.3.1). A schematic of the Interaction Hamiltonian can be seen in fig. 2.7.

2.3.1 Rapid adiabatic passage

A driving field is of interest to cause a specific population inversion in one or more atoms. To achieve this with a simple pulse is impractical since this would require a very finely tuned pulse length according to a phase shift of \( \pi \) in the atom’s Rabi oscillation.
Figure 2.7: Schematic of the interaction Hamiltonian with a driving field. $H_{int}^N$ represents the matrix (2.2.16) with excitation number $N$. Which is the first term in (2.3.1). The matrix $A$ represents the second term in (2.3.1).
A more robust way is with a Rapid adiabatic passage or RAP (11). The detuning $\Delta_d$ of the atom to the driving field is changed slowly from a negative to a positive value slow enough so the state vector can adiabatically follow. This can be achieved via a chirped pulse. A pulse becomes chirped when a pulse of the driving field undergoes a Doppler shift which slows down higher frequencies more than high frequencies. An Accoustooptical modulated (AoM) laser beam can be used too (12). This method used an acousto-optic modulator (13) which is a material that changes its optical properties under influence of the passing high frequency sound waves. Population inversions are possible with a range of non-linear effects. It just requires that the pulse starts with a low frequency and ends with a high frequency while scanning over resonance. In the simulation, this chirped pulse is implemented as going from $-\Delta_0$ to $+\Delta_0$ with $\Delta_0$ an arbitrary value large enough for there to be no effect of the driving field. During the RAP, the frequency changes as $(rt - \Delta_0)$ and sweeps over the resonance at $t = \Delta_0/r$.

The envelope of the pulse is implemented as a Gaussian with maximum at $t = \Delta_0/r$ the ends have no much importance since the large detuning means they have little effect on the further evolution of the system.

fig. 2.8 displays a simulation of a population inversion in one atom. The pulse reaches crosses the resonance at $t=0.025$.

### 2.4 The Tavis-Cummings Model for Arbitrary Amount of Atoms

The equations eq. (2.2.16) and eq. (2.2.15) are valid for any number of atoms. The amount of eigenvectors of $H_0$ is then $2^n$ which becomes unwieldy in their direct notation.

A computationally more practical way of handling the equations is to systematically label the eigenvectors of $H_0$ with a number. Besides the number state describing the photons, the combination of atom states too can be denoted with a number. If one labels the excited atoms $e = 1$ and the ground state atoms $g = 0$, then any combination of $e$ and $g$ resembles a binary number. For $N$ atoms, there are $2^N$ possible combinations. Such a number state can be written as follows.

$$|i\rangle = |a, n\rangle = |i\%N, i/2^N\rangle$$

(2.4.1)

In which $i/2^N$ denotes the photon state $n$ and $i\%n$ is the binary number represented by the excited or ground states of the atoms. The operator $\%$ is the modulo operator and the operator $/$ is a normal division which rounds to
Figure 2.8: Population inversion of a two level atom by a chirped pulse rapid adiabatic passage.
the lower integer. According to C-type language conventions. For example, in the case of 2 atoms, the state $|10\rangle$ has $10/4 = 2$ photons in the cavity and a binary number $10\%4 = 2$ which denotes the combination. e.g. The state $|ee0\rangle$ can therefore be written as $|3\rangle$.

With this notation the equations of motion for an arbitrary number of atoms with an arbitrary driving field can be given by:

$$
\Psi = \sum_i c_i |i\rangle = \sum_i c_i |i\%n = a, i/n >
$$

$$
\dot{c}_i = -i g_1 \sqrt{j/n + (j/n > i/n)} \exp\left(\frac{a-b}{|a-b|} i(\Delta t + k \cdot r_s)\right) c_j
$$

$$
+ \sum_{j'=(k=0)}^{j(n=n-1)} i g_2^k \exp\left(\frac{i a - b}{|a-b|} \cdot \Delta dt\right) c_{j'}
$$

This formula depicts the summation over all contributions of the states $|j\rangle$. These states consists of all states with an atom state cobination which differs 1 bit from the atom state of $|i\rangle$. This explains the first demand $j\%n = i\%n \oplus 2^k = b$ in which all states with the $k$th bit are changed. Hence the binary operator $\oplus$ which denotes an exclusive OR. This binary number with the $k$th bit changed is assigned the name $b$. The second condition under the summations specifies the photon number of the $|j\rangle$ states to be used. In the case of the normal cavity interaction or the first summation, this just assumes that $|j\rangle$ has to have the same excitation number of $|i\rangle$. In the second summation which depicts the contributions of a driving field, the $|j\rangle$’s are used with the same photon number as $|i\rangle$. These conditions filter out the $j$’s systematically so the system can be computationally integrated with one single function for an arbitrary number of atoms with an arbitrary number of photons. Wherein the indexes start from 0 and the operation that act on them are considered integer operations that approximate to the lower integer value. In the second term, the couplings of the atoms to the driving field is denoted with $g_2^k$. These are all 0 if there is no driving field, but if for example a population inversion is needed the $p$-th atom is set to a specific value.

2.5 **Pure N photon states**

One of the main reasons of interest in optical cavities is the production of pure N photon Fock states. It is worthwhile to examine the possibility of creating
such a Fock state with a single manipulation of the atoms. There are already practical realisations of n photon Fock state production that use the evolution of single atom in a cavity as described by the Jaynes-Cummings model but these are sequential excitation per excitation. With clever usage of the Rabi oscillation knowing that probability of the maximal photon number reaches its maximum at \( tg \sqrt{n} + 1 = k \pi \) and therefore the atom is in the ground state with near certainty, one can add an excitation to the system (by means of a population inversion of the atom) on just the right time so the system will evolve to a new pure photon state with one photon extra. A practical more practical realisation of this is described in the article (14) in which they don’t excite the atom, but pass pulses of excited atoms trough the cavity on the right time.

It would be interesting to see if a pure photon state can be achieved in one single step. For example in the case of 2 atoms, it is easy with RAP to produce a pure |ee0⟩ state, but as can be seen in section 2.2 which depicts the evolution of the system during one period of the 2-atom Rabi oscillation \( T = \frac{2\pi}{g^2} \) starting from a pure |ee0⟩ state. The evolution never reaches a probability 1 for the |gg2⟩ state. Secondly, at any point in time, the evolution of a system that reaches a pure |ee0⟩ state never reaches a vector with populations that are equal to any point in time of an evolution that does reach the state |gg2⟩. Therefore changing the phase of all coefficients is not possible unlike the 1-atom system.

If one considers the two states, in the basis spanned by the dressed states, then it becomes clear that in order to go from a |ee0⟩ to a |gg2⟩ state, the coefficients of the dressed states |1⟩ and |4⟩ must acquire a phase shift of \( \pi \) while those of the dressed states |2⟩ and |3⟩ must remain invariant.

\[
|g g, 2 > = \begin{pmatrix}
-1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & -1
\end{pmatrix} |e e, 0 >
\]

It is however not as easy to manipulate the dressed states of the system as it is to manipulate the bare states. However, if one looks at the eigenvalues accompanying the dressed states in fig. 2.5 one can see that the energies in function of the detuning \( \Delta \) of |2⟩ and |3⟩ are perfectly symmetric with reference to the y-axis, and the energies of |1⟩ and |2⟩ are not. It should therefore be possible to change the detuning \( \Delta \) symmetric about the y-axis in such a way...
that the dressed states $|1\rangle$ and $|2\rangle$ get a net phase change of $\pi$ while the net phase change of $|2\rangle$ and $|3\rangle$ is 0.
CHAPTER 3

LEAKING CAVITIES

3.1 PROPERTIES AND HAMILTONIAN

So far, only perfect cavities were considered in which the photons are trapped indefinitely. It has a transmission coefficient \( t \neq 0 \). This can be seen as a coupling of the EM field inside the cavity to the EM field outside the cavity. Such a system of an optical resonator coupled to free space via a semi-transparent mirror can be modelled by a Helmholtz equation and solved exactly. Unlike a closed cavity, the solution consists of a continuum of different frequency modes, the amplitude of the modes is generally called the spectral response function. (15)

\[
T(\omega) = \frac{t(\omega)}{1 + r(\omega) \exp(i \frac{2\omega}{c})}
\]

(3.1.1)

with \( l \) the length of the cavity For small ranges of frequency \( \omega \), the dielectric constant of the material of the mirror and therefore its transmission and reflection coefficient can assumed to be constant.

\[
|t(\omega)| = |t|
\]

(3.1.2)

\[
|r(\omega)| = |r|
\]

(3.1.3)

If one for example assumes that \( t \to 0 \), then the poles of eq. (3.1.5) will be the most prominent frequencies in the spectral response function.

\[
(1 + r \exp(\frac{il\omega}{c})) = 0
\]

(3.1.4)
These frequencies will be: $\omega_m = m \frac{\pi c}{l}$ with $m$ an integer number. These are the resonant frequencies of the cavity itself.

If one imagines the photon in the cavity bouncing back and forth against the cavity mirrors, it hits the partially transmissive mirror with a frequency $\frac{c}{2l}$ with $l$ the length of the cavity. And every time it has a probability of $|t|^2$ to be emitted. Therefore, one can say that the cavity has a decay rate:

$$\gamma = \frac{c}{2l} |t|^2$$  \hspace{1cm} (3.1.5)

The Hamiltonian of a leaking cavity is analogous to eq. (2.3.1). But instead of a coupling to a driving field, the electromagnetic field in the cavity is coupled to the electromagnetic field of the free space outside the cavity.

$$H = H_{cav} + H_{int} + H_{res}$$  \hspace{1cm} (3.1.6)

with $H_{cav}$ the Hamiltonian of the system inside the cavity. The system discussed here is the same two atom system as eq. (2.2.1). Since here systems with maximally 2 excitations are considered, the Hamiltonian is a block diagonal matrix with 3 blocks. One block for 2,1 and 0 excitations. Written out in matrix notation:

$$H_{cav} = \begin{bmatrix}
0 & g & g & 0 \\
g & 0 & 0 & \sqrt{2}g \\
g & 0 & 0 & \sqrt{2}g \\
0 & \sqrt{2}g & \sqrt{2}g & 0 \\
\end{bmatrix}$$  \hspace{1cm} (3.1.7)

$H_{res}$ the Hamiltonian of the reservoir which is the electromagnetic field of the free space outside the cavity. Unlike the the cavity field, the reservoir field consists of a continuous range of frequencies.

$$H_{res} = \hbar \int d\omega \omega c^\dagger(\omega)c^\dagger(\omega)$$  \hspace{1cm} (3.1.8)
wherein $c^\dagger(\omega)$ and $c(\omega)$ represent the creation and annihilation operators of the modes of the EM field with frequency $\omega$. Again the zero-point energy is omitted. And lastly the term $H_{\text{int}}$ which describes the coupling of the field inside the cavity to the field outside the cavity.

$$H_{\text{int}} = \hbar \int d\omega \kappa(\omega)(a^\dagger c(\omega) + a c^\dagger(\omega)) \quad (3.1.9)$$

### 3.2 The Langevin Equation

As proposed in (3) (2), it is possible to model the evolution of operators in an open cavity with a Langevin equation. To see how a Langevin equation can be obtained, one starts from the Heisenberg equations derived from the Hamiltonian eq. (3.1.6). The Heisenberg equation for an arbitrary operator $\hat{A}$ are:

$$\dot{c}(\omega) = -i \omega a(\omega) + \kappa(\omega)c(\omega) \quad (3.2.1)$$

$$\dot{A} = -\frac{i}{\hbar}[A, H_{\text{cav}}] + \int d\omega k(\omega) \left\{ c^\dagger(\omega) [A, a] - [A, a^\dagger] c(\omega) \right\} \quad (3.2.2)$$

eq. (3.2.1) can be substituted into eq. (3.2.2). The first Markov approximation is applied. This approximation assumes that the coupling constant is independent of $\omega$ which is evident since the coupling $\kappa$ originates in the partially transmissive mirror from which the properties are already assumed independent of the frequency eq. (3.1.2).

$$\kappa(\omega) = \sqrt{\frac{\gamma}{2\pi}} = \frac{k}{\sqrt{2\pi}} \quad (3.2.3)$$

The field outside the cavity consists of an incoming component $c_{\text{in}}$ travelling towards the cavity and an outgoing component $c_{\text{out}}$ travelling away from the cavity. $c_{\text{in}}$ the annihilation operator for the incoming component of the EM field outside the cavity can be defined as:

$$c_{\text{in}}(t) = \frac{1}{\sqrt{2\pi}} \int d\omega e^{-i\omega(t-t_0)} c_0(\omega) \quad (3.2.4)$$
with \( c_0(\omega) \) is \( c(\omega) \) at time \( t_0 \). By using eq. (3.2.4) in eq. (3.2.2) one can find the quantum Langevin equation:

\[
\dot{A} = -\frac{1}{\hbar} [A, H_{cav}] - \left( \frac{\gamma}{2} a + \sqrt{\gamma} c_{in}(t) \right) - \left( \frac{\gamma}{2} a^{\dagger} + \sqrt{\gamma} c_{in}^{\dagger}(t) \right) [a, A] \tag{3.2.5}
\]

If the cavity mode \( a \) then is used as the operator \( A \) in the Langevin equation, one gets:

\[
a = -\frac{1}{\hbar} [a, H_{cav}] + \frac{\gamma}{2} a - \sqrt{\gamma} c_{in}(t) \tag{3.2.6}
\]

with the term \( \frac{\gamma}{2} a \) the damping term and the term \(-\sqrt{\gamma} c_{in}(t)\) can be interpreted as a "Langevin Force" or the noise term. Besides vacuum fluctuations, the noise generally is from a thermal origin but for practical reasons it mostly is treated as pure white noise but with its number of quanta \( N \) depending on the temperature.

\[
< c_{in}^{\dagger}(t)c_{in}(t') > = N \delta(t - t') \tag{3.2.7}
\]

And

\[
N(\omega) = \frac{1}{e^{(\frac{\hbar \omega}{kT})} - 1} \tag{3.2.8}
\]

Which is the Bose-Einstein density of states which is typical for boson statistics.

A major problem with white noise is the assumption that its value is completely different at any point in time. This gives rise to problems with differential equations. White noise can be seen as the derivative of a non differentiable, random but continuous function, a so called Wiener process.(16) A well known example of a Wiener process is Brownian motion. In order to avoid the problems of the white noise, the differential equation is turned into an integral equation and the Wiener Process is used instead. To avoid confusion with
notation, the notation of C.W. Gardiner and M.J. Collett (3) is used. The Wiener process here is implemented as:

$$B(t, t_0) = \int_{t_0}^{t} c_{in}(t') dt'$$ (3.2.9)

It is also worth specifying that $B$ and $B^\dagger$ are distributed as a quantum Gaussian. Since it is an integration over white noise which can be seen as a summation over an infinite amount of independent random numbers. This integral must be distributed by a Gaussian as dictated by the central limit theorem. This means the density operator of the noise $\rho(t, t_0)$ can be given by:

$$\rho(t, t_0) = (1 - e^{-\sigma}) e^{-\frac{\sigma B^\dagger(t, t_0) B(t, t_0)}{t - t_0}}$$ (3.2.10)

with $\sigma = \frac{\hbar \omega}{k_B}$. It can be seen that for low temperatures $T \to 0$ that $N \to 0$

Using eq. (3.2.7) and eq. (3.2.9) it is readily found that:

$$<B^\dagger(t, t_0)B(t, t_0)> = N(t - t_0)$$ (3.2.11)

Which indeed is similar to Brownian Motion in which the quadratic distance is proportional to the time.

There are two standard methods of quantum stochastic integration. Ito integration and Stratanovich integration. The former has the advantage that the calculus is far less complicated than Stratanovich calculus. but only Stratanovich integration treats the non-commuting calculus of quantum operators correctly. The quantum stochastical differential equation in Stratanovich form can then be found as:

$$dA = -\frac{i}{\hbar} [A, H_{cav}] dt - \frac{\gamma}{2} \left( [A, a^\dagger] a - a^\dagger [A, a] \right) dt$$

$$- \sqrt{\gamma} \left[ A, a^\dagger \right] dB(t) + \sqrt{\gamma} dB^\dagger(t) [A, a]$$ (3.2.12)

As stated before, as the Langevin equation was derived in the Heisenberg picture, this QSDE describes the evolution of operators. Since the evolution of the coefficients or populations of the different cavity states is more of interest, the QSDE is mainly used to derive the more practical Master Equation.
3.3 The master equation

The cavity leaks its photons into free space, this can be seen as a system in contact with a reservoir. The decoherence of a system is modeled most commonly via a Master Equation which describes the evolution of the density operator $\hat{\rho}$. The system and reservoir are considered one as one large system. The master equation is obtained after tracing out the degrees of freedom of the reservoir. Using the QSDE eq. (3.2.12) as described in (3) one can find:

$$\frac{d\rho}{dt} = \frac{i}{\hbar}[\rho, H_{sys}] + \frac{\gamma}{2}(N + 1)(2a\rho a^\dagger - a^\dagger a \rho - \rho a^\dagger a)$$

with $N$ still the number of quanta of the output field. In the case of $N = 0$ (at $T = 0$) one can easily recognize the Lindbladian form of the Master equation. $N \neq 0$ can be interpreted as the open system being in contact with a heat bath that generates noise which influences the overall evolution of the system. The atoms in the cavity are considered two-level atoms and can decay spontaneously which too can be described via a coupling of the atom to the EM field of free space. This in turn can be described by a master equation. The same is true for the inefficiencies in the cavity mirrors. Realistic mirrors not only transmit and reflect, but also partially absorb photons. This too can be described with a master equation. All sources of leakage and decoherence can each contribute a Lindbladian term to the total master equation. Here however, only the leakage of the cavity through the mirrors is discussed and the other decay paths are assumed to be negligible.

An example of the evolution of the populations (the diagonal elements of the density operator $\rho$) of the different states of 2 atoms in a leaky optical cavity can be seen in fig. 3.1. The system starts from two excited atoms and zero photons in the cavity ($|ee0\rangle$) and starts evolving with a 2-atom Rabi oscillation similar to section 2.2. If the cavity does not contain atoms, a two photon state $|2\rangle$ will decay exponentially into the state $|1\rangle$ which on its turn will decay exponentially in the ground state $|0\rangle$. But since there are two atoms in the cavity, the probability of finding photons in the cavity evolves periodically. This can be seen most easily be observed by the wavy increase of the population of the ground state. The times when the ground state population is nearly constant coincide with a high probability of finding one or two atoms excited. In the case that the system starts its evolution with one excited atom $|eg1\rangle$ or
Figure 3.1: The evolution of the populations of cavity states as described by the Master Equation starting from a pure $|ee0\rangle$ state.
|eg0⟩ or the other atom, the cavity does not decay completely but is left with an asymmetric superposition of the two states with an excited atom. This is discussed in section 3.6.

### 3.4 Quantum Jump Operator Method

An alternative to the master equation is the quantum jump operator method. This is a Monte-Carlo simulation of the system which is equivalent to the master equation. (17) If there are photons in the cavity, there is a probability that in the following time step dt a photon is leaked out of the cavity. dt is assumed to be small enough that the probability of two simultaneously leaked photons can be neglected. It is already established that the leaking can be described via a damping coefficient γ. The probability for a photon to be emitted in a time step dt is then:

\[ P = \gamma \langle \Psi | a a^\dagger |\Psi \rangle \]  

(3.4.1)

with |Ψ⟩ the state vector of the system. As this is a Monte Carlo method, a random number r is generated and compared with the probability. If \( r < P \), the system emits a photon which is described with a lowering operator.

\[ |\Psi⟩ → \frac{a |\Psi⟩}{\langle \Psi | a a^\dagger |\Psi⟩^{1/2}} \]  

(3.4.2)

If however \( r > P \) no photon is emitted and the evolution of the cavity continues with the same number of excitations. Since knowledge of the absence of an emitted photon influences the state in the cavity, the system will evolve according to a non-Hermitian effective Hamiltonian \( H_{eff} \) which is not exactly the same as eq. (2.2.2). (17)

\[ H_{eff} = H_{sys} - i \hbar \frac{\gamma}{2} a a^\dagger \]  

(3.4.3)

Therefore, in a timestep \( dt \), the evolution of the system according to this Hamiltonian can be given by:

\[ |\Psi⟩ → \frac{1 - dt \frac{\gamma}{\hbar} H_{eff} |\Psi⟩}{(1 - P)^{1/2}} \]  

(3.4.4)
A renormalisation is necessary since the evolution is described by a non-hermitian Hamiltonian and the particle is not detected with a probability $(1 - P)$. This evolution is then repeated multiple times and the average of all evolutions converges to the evolution described by the master equation. In following fig. 3.2 and fig. 3.3, the averages over 100 and 1000 runs can be seen for the same decay constant $\gamma = \frac{3}{4} g$. With $g$ still the coupling of the atoms to the cavity field. The value of $\gamma$ is arbitrarily chosen so that the photons do not escape the cavity in a short time, but long enough to clearly see a pattern in the detection times of the photons which will be discussed in the next paragraph. In the case of 100 runs, the effects of the separate quantum jumps are still visible in the average evolution. In the case of 1000 runs, the evolution smoothens out till it matches the master equation.

The main advantage of the quantum-jump operator method over the master equation is the fact that each run imitates a real experiment and is therefore far more flexible when it concerns the collection of data. E.g. the times at which the first and second photon are detected can be stored separately which is much harder to do when one considers only the Master equation. And histogram of the times when a photon is emitted (the system jumps) can be seen in fig. 3.4. Since the system starts from two excited atoms, there are no photons in the cavity at $t = 0$ hence no photons are emitted. As the system evolves with a 2-atom Rabi frequency, the probability of finding a photon in the cavity increases so does the probability detecting an emitted photon. The pattern in the beginning follows the evolution of the system with two excitations. However, if a photon is emitted, the cavity is left with only 1 excitation. The two-atom Rabi frequency with one photon is $\sqrt{2} g$. So one expects a frequency shift which is not directly apparent in fig. 3.4. But one must take into account that the probability is the square of the coefficient and the square of a (co)sine wave appears to have a double frequency. (the same effect is noticeable in the populations of the states $|eg1\rangle$ and $|ge1\rangle$ in section 2.2.) The new apparent 1-excitation period $\frac{\pi}{\sqrt{2}g} \approx 2.22$ is slightly shorter than the first 2-excitation period $\frac{2\pi}{\sqrt{6}g} \approx 2.56$. Therefore a slight frequency shift in the pattern of detected photons is predicted. Although not statistically significant in fig. 3.4, this will be discussed further in the end of the following section.

An other interesting possibility of the quantum jump operator method is the possibility of collecting the time difference between first and second detected photons. A histogram of which can be seen in fig. 3.5. Since the photon starts...
Figure 3.2: The average evolution of 100 runs in the quantum jump operator method.

from two excited atoms, it will evolve to a high probability of the state $|g g 2 \rangle$. Therefore, if a photon is emitted, the cavity has a high probability of being in the state $|g g 1 \rangle$ which easily emits the second atom. Therefore, it is very probable that the two photons are emitted shortly after one other. If not, the cavity evolves with the 1-excitation 2-atom Rabi frequency as described in the previous chapter till the probability of finding a photon in the cavity is high again. Therefore, the waveform visible in fig. 3.5 should have a constant frequency $\sqrt{2}g/2$. 
Figure 3.3: The average evolution of 100 runs in the quantum jump operator method.
Figure 3.4: Histogram of 1000 runs in 100 bins of the times when photons are emitted from the cavity.
Figure 3.5: Histogram of 1000 runs in 100 bins of the time difference between the emission of the first photon and the second photon.
Figure 3.6: Schematic of a leaking cavity with two photons. As the photon density exits the cavity, it travels with speed \( c \)

### 3.5 Modelling the outgoing photon

The Master equation and the quantum-jump operator are mainly used to describe only the evolution of the degrees of freedom of the open system. The Master equation method assumes a trace over the reservoir, while the quantum-jump Monte Carlo method only provides the instances when the system "jumps" and emits a photon. These methods assume that the photon interacts directly with its environment and are not sufficient if one wants to take in account the photons after they leak out of the cavity.

This chapter aims to find a method of integrating the system that explicitly includes the photon density directly outside the cavity, since at any given time there will be a superposition of 0, 1 or 2 photons outside the cavity.

This modelling of the photon outside the cavity has the problem that when the photon is emitted in free space, it will travel away with the speed of light. Therefore, this simulation splits up the outgoing photon in spatio-temporal modes. This means that the space outside the cavity is divided into bins, and after each small time interval \( dt \), the bins are shifted a distance \( c dt \). And each time, the cavity leaks some photon density in a new empty bin which gets shifted along in this "conveyor belt". This mimics the movement of the escaped photon wave. This simulation will only consider the first bin. In chapter 4 the coherent evolution of multiple large bins (low resolution) will be simulated.

The total Hamiltonian of the entire system is considered. The Hamiltonian is again the same as eq. (2.2.1) but with two additional terms, the energy of
the vacuum, and the coupling of the cavity to the vacuum. (again, the phase factors of the position $e^{ik\cdot r_i}$ are left away.) The free space outside the cavity contains a continuous range of frequencies over which must be integrated to include all contributions.

$$H = \hbar \omega a^\dagger a + \frac{1}{2}\hbar \omega \sigma_1^+ + \frac{1}{2}\hbar \omega \sigma_2^+ + \hbar \int d\omega \; c^\dagger(\omega) c(\omega)$$

$$+ \hbar g \left( a\sigma_1^+ + a\sigma_2^+ + a^\dagger\sigma_1^- + a^\dagger\sigma_2^- \right)$$

$$+ \hbar \int \kappa(\omega) \left( a^\dagger c(\omega) + ac^\dagger(\omega) \right)$$

(3.5.1)

In an equivalent way as in chapter 2 and 3, one can move to the interaction picture. The integral in the last term, can be regarded as a sum with infinite amount of terms with each a different frequency $\omega_v$.

And because the creation and annihilation operators of different frequencies commute, $[c(\omega_1), c^\dagger(\omega_2)] = \delta(\omega_1 - \omega_2)$

$$e^{i\omega_v c^\dagger(\omega_1)c(\omega_2)t} c(\omega_1) e^{-i\omega_v c^\dagger(\omega_1)c(\omega_2)t} = c(\omega_1) e^{-i\omega_v t}$$

(3.5.2)

Therefore, the interaction Hamiltonian can be written as:

$$\mathcal{H}_{int} = \hbar g \left( a\sigma_1^+ + a\sigma_2^+ + a^\dagger\sigma_1^- + a^\dagger\sigma_2^- \right)$$

$$+ \hbar \int d\omega_v \kappa(\omega_v) \left( a^\dagger c(\omega_v)e^{i(\omega - \omega_v)t} + ac^\dagger(\omega_v)e^{-i(\omega - \omega_v)t} \right)$$

(3.5.3)

Again, the First Markov approximation is applied eq. (3.2.3). Whereupon again term can be seen as a Fourier transform so the integral can be written as a time dependency:

$$\mathcal{H}_{int} = \hbar g \left( a\sigma_1^+ + a\sigma_2^+ + a^\dagger\sigma_1^- + a^\dagger\sigma_2^- \right) + \hbar k(a^\dagger c(t) + ac^\dagger(t))$$

(3.5.4)

The properties of the noise creation and annihilation operators $c^\dagger(t)$ and $c(t)$ are already explained in section 3.2 A thing to be consider is how noise can be implemented. Similar to the case of the Langevin equation in the Heisenberg picture, the noise here is handled with its Wiener process. This Wiener process has again a random density produced by the very same quantum Gaussian
as 3.2.10. But with \((t - t_0) = d t\) This random photon density coming from
the noise can be replaced each time interval \(d t\). And assuming very low tem-
peratures, just like before, section 3.2 \(N \rightarrow 0\) with \(N\) again the same eq. (3.2.8).

Since the system and the first spatio-temporal bin outside the cavity is consid-
ered, the system can assumed to be closed for a short time \(d t\) as if the first
bin is actually a second cavity. The bare state vectors of this system consist
of two atom states and two number states \(|a_1a_2n_{in}, n_{out}\rangle\) with \(a_1\) and \(a_2\) the
state of the atoms just like before and \(n_{in}\) the number state of the photon in-
side the cavity, and \(n_{out}\) the photon state directly out of the cavity in the first bin.

In these temporally closed systems, a general state of the complete system can
be:

\[
\Psi = \sum_{a_1,a_2=\pm,\emptyset} C_{a_1a_2n_{in};n_{out}} |a_1a_2n_{in};n_{out}\rangle (3.5.5)
\]

The physical meaning of the kets are that the atoms can be excited or not, there
is a \(n_{in}\)-photon Fock state in the cavity and a (temporary) \(n_{out}\)-photon Fock
state in the first bin outside the cavity.

The equations of motion in the interaction picture in such a short timespan \(d t\)
are again:

\[
i\hbar \frac{\partial \Psi}{\partial t} = \mathcal{H}_{int} \Psi \quad (3.5.6)
\]

To see how damping emerges in this system, no atoms are considered at first.
Initially, the volume outside the cavity contains 0 photons, therefore \(n_{out} = 0\).
According to the Hamiltonian eq. (3.1.6) a simple cavity with \(n\) photons and
no atoms can be described with the equations of motion.

\[
\begin{align*}
\dot{C}_{n,0} &= i k \sqrt{n} \sqrt{1} C_{n-1,1} \\
\dot{C}_{n-1,1} &= i k \sqrt{n} \sqrt{1} C_{n,0}
\end{align*} \quad (3.5.7)
\]

In which the notation \(C_{n_{in},n_{out}}\) is used with \(n_{in}\) depict the photons in the cavity
and \(n_{out}\) the photons outside the cavity. \(k\) is again the coupling of the field
inside the cavity to the outside. The equations simply depict an ordinary
Harmonic oscillator with \(C_{n,0}(t) = C_{n,0}(t_0) \cos t\) and \(C_{n,0} = C_{n,0}(t_0) \sin t\) The
system evolves for a short period \(d t\) after which a small amount of probability
accumulates for the photon to be outside the cavity and the cavity has one
excitation less. To model the flying away of the photon, the coefficient \(C_{n-1,1}\)
is then set to 0 after being added Pythagoreancally to the coefficient $C_{n-1,0}$. This Pythagoreanical addition is simply an addition that conserves the norm of the system and also takes the phase in account. One cannot simply add the coefficients.

This effectively mimics a shift of the photon density over a distance $c d t$ so a new area of space can be filled. Then the system evolves again a time $d t$ and the photon density is shifted again.

Since each of these steps takes an extremely short period of time, it is better to approximate the solutions of eq. (3.5.7) by their Taylor expansion to first order. With $\dot{C} \approx \frac{dC}{dt}$ the evolution of the cavity in small discrete steps can be described with:

\[
\begin{align*}
\frac{dC_{n,0}}{dt} &= C_{n,0}(t) \frac{\cos(t+dt) - \cos t}{dt} \approx -\frac{k^2}{2} n C_{n,0}(t) \\
\frac{dC_{n-1,1}}{dt} &= C_{n,0}(t) \frac{\sin(t+dt) - \sin t}{dt} \approx k \sqrt{n} C_{n,0}(t)
\end{align*}
\] (3.5.8)

which is again compatible with the effective Hamiltonian as described in eq. (3.4.3). And again the damping factor $\gamma = k^2$ is found similar to the Langevin equation. These equations do indeed predict an exponential decay of the photons in such a cavity. As can be seen in the second equation of eq. (3.5.8) and similar to (2), the amplitude of the outgoing field directly outside the cavity is $k \sqrt{n} C_{n,0}(t)$. Therefore, the probability of finding a photon in the first bin after a time $d t$ can then be given by $n k^2 C_{n,0}^*(t) C_{n,0}(t) d t$. The factor $d t$ comes from the fact that this first bin had a time $d t$ to fill.

The equations of motion given in eq. (3.5.9) are simply the equations of motion as given in eq. (2.2.3) but for respectively 2 and 1 excitation and with a damping factor for each coefficient of states with one or more photons in the cavity as given by eq. (3.5.8).
\[ 
\dot{C}_{ee,0,0} = g \left( C_{ge,1,0} e^{i\Delta t} + C_{eg,1,0} e^{i\Delta t} \right) 
\] 
(3.5.9)

\[ 
\dot{C}_{eg,1,0} = g \left( C_{ee,0,0} e^{-i\Delta t} + \sqrt{2} C_{gg,2,0} e^{i\Delta t} \right) - \frac{\gamma}{2} C_{eg,1,0} 
\] 
(3.5.10)

\[ 
\dot{C}_{eg,1,0} = g \left( C_{ee,0,0} e^{i\Delta t} + \sqrt{2} C_{gg,2,0} e^{-i\Delta t} \right) - \frac{\gamma}{2} C_{ge,1,0} 
\] 
(3.5.11)

\[ 
\dot{C}_{gg,2,0} = g \left( \sqrt{2} C_{ge,1,0} e^{-i\Delta t} + \sqrt{2} C_{eg,1,0} e^{i\Delta t} \right) - \gamma C_{gg,2,0} 
\] 
(3.5.12)

\[ 
\dot{C}_{eg,0,0} = g \left( C_{gg,1,0} e^{i\Delta t} \right) 
\] 
(3.5.13)

\[ 
\dot{C}_{ge,0,0} = g \left( C_{gg,1,0} e^{i\Delta t} \right) 
\] 
(3.5.14)

\[ 
\dot{C}_{gg,2,0} = g \left( C_{ge,1,0} e^{-i\Delta t} + C_{eg,1,0} e^{-i\Delta t} \right) - \frac{\gamma}{2} C_{ge,1,0} 
\] 
(3.5.15)

\[ 
\dot{C}_{ge,0,0} = g \left( C_{gg,1,0} e^{-i\Delta t} \right) 
\] 
(3.5.16)

Similarly for the coefficients of the states with a photon outside;

\[ 
\dot{C}_{eg,0,1} = k C_{eg,1,0} 
\] 
(3.5.17)

\[ 
\dot{C}_{ge,0,1} = k C_{ge,1,0} 
\] 
(3.5.18)

\[ 
\dot{C}_{gg,1,1} = \sqrt{(2)} k C_{gg,2,0} 
\] 
(3.5.19)

\[ 
\dot{C}_{gg,0,1} = k C_{gg,1,0} 
\] 
(3.5.20)

But depending on their use, these last four equations are not necessary, since these coefficients can be expressed in terms of the coefficients of eq. (3.5.9). But they allow for a direct manipulation of the photons outside the cavity.

These equations are integrated for a small time step \( dt \). After this time step, the photon density in the first bin outside the cavity will move away and therefore leave the system with one excitation less. For example: \( C_{eg,0,1} \rightarrow C_{ge,0,0} \). Therefore, the probability of finding a photon in this first bin, must be added to the equivalent state without the photon in that bin. The main requirement for \( dt \) is that it has to be small enough to be able to neglect second order effects such as the leaking of the state \( C_{gg,1,1} \) to \( C_{gg,0,2} \) or its evolution into the states \( C_{ge,0,1} \) and \( C_{eg,0,1} \). The coefficients change in the following manner:

\[ 
C_{eg,0,0} = \sqrt{C_{eg,0}^2 + \gamma C_{eg,1}^2} dt 
\] 
(3.5.21)

\[ 
C_{ge,0,0} = \sqrt{C_{ge,0}^2 + \gamma C_{ge,1}^2} dt 
\] 
(3.5.22)

\[ 
C_{gg,1,0} = \sqrt{C_{gg,1}^2 + 2\gamma C_{gg,2}^2} dt 
\] 
(3.5.23)

\[ 
C_{gg,0,0} = \sqrt{C_{gg,0}^2 + \gamma C_{gg,1}^2} dt 
\] 
(3.5.24)
Notice that a normal square is used instead of a product with its hermitian conjugate. This is to incorporate the phase in the addition. This is why this summation was called a “Pythagorean” addition earlier. Now the coefficients are reset and can be used again for the following time step of $dt$.

The most desirable property of this model is that it gives direct access to the superposition of photon number states outside the cavity instead of only a density. This property will be used in the following chapter to detect 0, 1 or 2 photons and collapse the wave function of the cavity accordingly.

The evolution of the populations of the different cavity states matches up with the Master equation and the quantum jump operator method and is not plotted to avoid a third similar plot. A more interesting plot is the photon density leaking out of the cavity which is plotted in fig. 3.7. This figure is equivalent to the histogram fig. 3.4 of the detection times of the photons in the quantum jump operator method. In the discussion of that figure, it is explained that there is a frequency shift expected. In fig. 3.7 this frequency shift can be seen. The period of the first peak (2.50 from minimum to minimum) coincides roughly with the period of the 2-excitation 2-atom Rabi Frequency. The frequency is slightly off since a large proportion of the photon density has already leaked out of the cavity before the end of the first peak. The period of the fifth peak (2.250 from minimum to minimum) matches almost perfectly with the period of a 1-excitation Rabi frequency as predicted.

### 3.6 Entanglement by Dissipation

Up to now, the evolutions of the system have always started from two excited atoms. $\Psi(t_0) = |ee0\rangle$. The system can equally well begin with initially only one excited atom. E.g. the state $\Psi(t_0) = |ge0\rangle$. In this case however, the system behaves in a surprising way. The excitation in the system decays only partly and the system remains partly in a superposition of the states $|ge0\rangle$ and $|eg0\rangle$. This behaviour can be seen in fig. 3.8.

Due to the fact that the cavity is damped, its spectrum broadens and becomes continuous. Weisskopf-Wigner theory then dictates that an atom coupled to a continuous range of frequencies decays spontaneously (18). And indeed, the simulation of a single atom coupled to a damped cavity does indeed decay completely to its ground state. Here however, two atoms are considered and there are some complications. Consider the equations of motion in a closed cavity eq. (2.2.3) for one excitation.
Figure 3.7: The photon density directly outside the cavity in function of time for a leaking cavity with 2 atoms and decay constant $\gamma = \frac{3}{4}g$
Figure 3.8: The photon density directly outside the cavity in function of time for a leaking cavity with 2 atoms and decay constant $\gamma = \frac{3}{4} \gamma$. 
\[ \begin{align*}
\dot{C}_{eg,0} &= -igC_{gg,1} \\
\dot{C}_{ge,0} &= -igC_{gg,1} \\
\dot{C}_{gg,1} &= -igC_{ge,0} - igC_{ge,0}
\end{align*} \] (3.6.1)

The position depended phases are omitted, since they have no effect on the actual evolution of the populations of the states. It can be seen that \( \dot{C}_{gg,1} = 0 \) in the case that \( C_{eg0} - C_{ge0} = 0 \). Secondly, a quantum state \( \Psi = \frac{1}{\sqrt{2}} |ge0\rangle - \frac{1}{\sqrt{2}} |eg0\rangle \) (with of course \( C_{gg1} = 0 \)) is an eigenvector of the equations of motion in the interaction picture.

Since the damping of the cavity only has an effect on the Fock state of the cavity, the amplitude of \( C_{gg1} \) will decay accordingly. The shrinkage of \( C_{gg1} \) effectively means that the oscillation of \( C_{ge1} \) and \( C_{eg1} \) will shrink to a stationary state. And since the two coefficients in this type of evolution always have a different sign the stationary state will be achieved if \( C_{eg0} + C_{ge0} = 0 \). Which happens when their absolute values are equally large. If the position dependent phases are left in the equation, the same condition still applies.

In the case the evolution starts with \( \Psi t_0 = |ee0\rangle, |gg2\rangle \) or \( |gg1\rangle \) the coefficients \( C_{ge1} \) and \( C_{eg1} \) always have the same sign at any given time. Therefore, fulfilling the condition \( C_{eg0} + C_{ge0} = 0 \) for a stationary state means that \( C_{eg0} = C_{ge0} = 0 \) and that the cavity will decay to the ground state completely.

Although realistic systems provide more channels for the atoms and cavity to decay, this simple system shows that in theory one can produce an entangled state just letting the system decohere. If one doesn’t observe a photon exiting the cavity, the cavity will contain two pure entangled atoms.

Entanglement by dissipation is known already. For example in the case of two nearby atoms that affect each other with a dipole interaction a very similar thing happens. (19) (20) Here, the dipoles of the atoms don’t interact with each other, but with an optical cavity mode. So the very same mechanism applies, the cavity mode becomes entangled with the first atom and the second atom becomes entangled with the cavity. The cavity mode decays and what is left are two entangled atoms.
CHAPTER 4

EXPLICIT MODELLING OF LEAKED PHOTON DENSITY

The dynamics of open systems such as leaky optical cavities are generally described by a master equation. In the case of an optical cavity, this assumes that the dissipated photons immediately interacts with the environment\[1\]. If however a detector is placed at a considerable distance from the cavity, the emitted photons do not immediately interact with it. Instead the system keeps evolving coherently for some time depending on the distance of the cavity. It is not directly obvious that the same master equation can be used to describe the entire system of both the cavity and photon in the space between the cavity and the detector.

Therefore the cavity and the travelling photon are modelled explicitly with a detector on different distances to examine if this distance has any effect on the evolution of the system. Again, the emitted and moving photon wave is modelled with coarse spatio temporal bins. The bins have a size that reflects the resolution of the detector. With the aid of the method described in section 3.5 one can accumulate the photon densities in the large bins. As the bin then is read by the detector, it will have a well defined probability of detecting 0,1 or 2 photons and thereby project the wave of the cavity and the not-yet detected part of the photon.

4.1 HOW THE COHERENT EVOLUTION IS MODELED

Again, a system of 2 atoms in a resonant, leaking cavity is considered. A detector at some distance outside the cavity can detect the photons after they have left the cavity. The system evolves in very much the same way as described in section 3.5. here
however, much larger spatio-temporal bins are used, first of all to simulate a photon-counting detector with a coarse resolution, and secondly to reduce the simulation time. The distance $D$ between the detector is divided into $m$ bins. The system begins evolving from for example a pure $|ee0\rangle$ state. The photon density radiating from the cavity, will travel the distance of the first bin in a time $\frac{D}{cm}$. During this time, there is a probability that the cavity has emitted 0,1 or 2 photons in this first bin. The cavity is respectively left with 2,1 or 0 excitations, these cavity states are respectively denoted with $S_2$, $S_1$ and $S_0$. For example $S_1$ is just a linear combination of the states $|eg0\rangle$, $|ge0\rangle$ and $|gg0\rangle$. This simulation treats each possibility separately.

Schematicly, the first step can be seen as this:

$$S_2 \rightarrow \begin{cases} S_2; 0 \\ S_1; 1 \\ S_0; 2 \end{cases}$$

Wherein the number after the $;$ designates the number of photons emitted in the first bin. The contents of the first bin are then shifted to the second bin and the cavity system is evolved again. If the system still contained 2 excitations, there will again be a chance that it emitted 0,1 or 2 photons. If it had one excitation left, it only has a probability of emitting 0 or 1 photons. If the systems has no excitations left, it will be in the ground state and not evolve any further and therefore emit 0 photons. After the second step, the system will have been divided up like this:
Notice that there are now two "filled" bins, separated by a . If there are 3 bins, the system will again evolve and split up analogously as the previous steps.

This evolution continues depending on how much bins there are considered between the cavity and the detector. For simplicity, there are only 3 bins considered in this explication. Therefore, in the next step, the first emitted bin now reaches the detector. The last number in all the states (represented in bold) now represents that first bin shifted 2 places.

The detector will now detect 0 photons with a probability:

\[ P(S_2; 0, 0, 0) + P(S_1; 1, 0, 0) + P(S_0; 0, 0, 0) \]

The detector will detect 1 photon with a probability:

\[ P(S_1; 1, 0, 0) + P(S_0; 0, 0, 0) \]
\[ P(S_1; 0, 0, 1) + P(S_0; 1, 0, 1) + P(S_0; 0, 1, 1) \]

And finally, the detector will detect 2 photons with a probability:
\[ P(S_0; 0, 0, 2) \]

As an example for demonstration, it is assumed that the detector detects 1 photon. Then the entire system is projected on this observation. And it is evolved again in exactly the same way. All the bins are again shifted to the right.

\[
S_2 \rightarrow \left\{ \begin{array}{l}
S_2; 0, 0 \rightarrow \{ S_2; 0, 0, 0, S_1; 0, 0, 0, S_0; 2, 0, 0 \} \\
S_1; 1, 0 \rightarrow \{ S_1; 0, 1, 0, S_0; 1, 1, 0 \} \\
S_0; 2, 0 \rightarrow \{ S_0; 0, 2, 0, S_1; 0, 0, 1, S_0; 1, 0, 0 \} \\
S_1; 0, 1 \rightarrow \{ S_0; 1, 0, 1, S_0; 0, 1, 0 \} \\
S_0; 1, 1 \rightarrow \{ S_0; 0, 1, 1, S_0; 0, 0, 1 \} \\
S_0; 2 \rightarrow \{ S_0; 0, 2, 0, S_0; 0, 0, 2 \}
\end{array} \right.
\]

Notice that there are still 3 numbers behind the ; this is because here only 3 bins are considered and as one bin is detected, the bin just outside the cavity is filled up again. Secondly, notice that as one photon is already detected, only states with one excitation in the cavity or one photon in the bins are left. If no bins are between the detector and the cavity, it can be seen that this algorithm turns into an ordinary, yet very course "quantum-jump Monte-Carlo" method. The very same algorithm can be seen schematically in fig. 4.2. There, the probabilities are more or less represented by the size of the different rectangles. The number of excitations left in the cavity is represented with a colour code: Blue represents...
2 excitations left in the cavity, green 1 and orange means the cavity itself is in the ground state.

4.2 The results

The algorithm described in the previous section is used in the simulation. Since each bin divides the system up more and more, the maximum number bins considered is 5. The bin size is chosen so that the time the light takes to traverse the distance of one bin \( \frac{D}{cm} \) is about the period of a two-atom Rabi-oscillation of the system. As one can see in fig. 3.7, one bin will contain about one "bulge" of the photon density. The coupling to the vacuum is assumed equal to the coupling of the atoms to the cavity. This makes sure that the cavity empties itself nearly completely in 5 bins.

The simulation is then run 400 times and each run, continues till 5 bins have been read by the detector. The average amount of detected photons per bin is given in table 4.1 for a detector at position "1" (directly adjacent to the cavity), 2 and 5. It can first be seen that these averages of detected photons per bin can be considered the same for all distances. The distance of the detector seems to have no effect on the evolution of the cavity.

The average photons detected per bin matches roughly the photon densities exciting the cavity as simulated with the quantum jump method or the method described in section 3.5. Which is in agreement with what ones should expect from the master equation.

Secondly, the simulation is done for a much smaller resolution. The bin size is now considered the distance the light travels in \( \frac{1}{4} \) of the the 2-atom Rabi oscillation. This is simulated again for 400 times for distances of 0,1 or 2 bins between the cavity and the detector. Again it can be seen that the detector has no effect on the evolution of the cavity. And again, although represented with only 5 numbers, a bulge of the photon density can be recognised.

This section concludes with the fact that this model predicts no effect of the distance that a detector is placed from the cavity.

<table>
<thead>
<tr>
<th>position (( Tc/5 ))</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1.30 ± 0.04</td>
<td>0.41 ± 0.03</td>
<td>0.19 ± 0.03</td>
<td>0.07 ± 0.01</td>
<td>0.023 ± 0.006</td>
</tr>
<tr>
<td>2</td>
<td>1.33</td>
<td>0.43</td>
<td>0.16</td>
<td>0.0025</td>
<td>0.0012</td>
</tr>
<tr>
<td>5</td>
<td>1.25 ± 0.05</td>
<td>0.47 ± 0.05</td>
<td>0.2 ± 0.03</td>
<td>0.065 ± 0.02</td>
<td>0.01 ± 0.007</td>
</tr>
</tbody>
</table>

Table 4.1: Average amount of photons detected in the 5 first consecutive bins. With a binsize of 1 rabi oscillation period.
Figure 4.2: A schematic representation of the simulation. Each bin represents a distance $D/cm$.

4.3 MACH-ZEHNDER-INTERFEROMETER

As described in previous section, the distance of the detector does not influence the evolution of the cavity or the shape of the light wave packet. It would be interesting to see how the light wave behaves if a Mach-Zehnder-interferometer is placed between the cavity and the detector. And if again the wave is modelled with spatio-temporal bins throughout the interferometer with both photons a different number of bins. This makes the wave front interfere with a part of itself emitted later in time. This will make the evolutions with a different excitation number left inside the cavity interfere. For example, the path with one photon outside the cavity and one excitation left inside the cavity can interfere with a path that has two photons already emitted and no excitations left in the cavity. It is unclear how the detectors on both exits of the Mach-Zehnder-interferometer will collapse the cavity and wave front as
they detect or not detect photons and it would be interesting to see if it has any effect on the evolution of the system.

The course of this simulation will be far more complex than the simple situation wherein only the distance between the detector and the cavity is considered. Here, separate arrays will have to be stored with different superposition of the system which will have to interfere on different places. Although schematically worked out, this has not been implemented in code due to time constraints.

<table>
<thead>
<tr>
<th>position (Tc/5)</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0.156 ± 0.03</td>
<td>0.62 ± 0.05</td>
<td>0.36 ± 0.04</td>
<td>0.16 ± 0.03</td>
<td>0.11 ± 0.03</td>
</tr>
<tr>
<td>1</td>
<td>0.13 ± 0.02</td>
<td>0.53 ± 0.04</td>
<td>0.45 ± 0.03</td>
<td>0.13 ± 0.02</td>
<td>0.13 ± 0.02</td>
</tr>
<tr>
<td>2</td>
<td>0.14 ± 0.03</td>
<td>0.61 ± 0.05</td>
<td>0.42 ± 0.04</td>
<td>0.095 ± 0.02</td>
<td>0.16 ± 0.03</td>
</tr>
</tbody>
</table>

Table 4.2: Average amount of photons detected in the 5 first consecutive bins. With a binsize of 1/5 rabi oscillation period.
CHAPTER 5

CONCLUSION

A system with two atoms in an optical resonant cavity experiences a 2-atom Rabi oscillation similar to the familiar single-atom Rabi oscillation from the Jaynes-Cummings model. However, unlike the single-atom case, if the system starts from two excited atoms $|ee0\rangle$ it never reaches a pure photon number state $|gg2\rangle$.

Those two states are examined in the base spanned by the bare states of this two-atom system, it can be seen that the evolution reaching periodically a pure $|ee0\rangle$ state and the evolution periodically reaching a pure $|gg2\rangle$ state differ only a phase $\pi$ for two coefficients in the dressed state base while the two other coefficients are the same.

As described in section 2.5 a phase change of $\pi$ for only those two coefficients can be obtained because the eigenvalues or energies of those dressed states are not symmetric with respect to $\Delta = 0$. Therefore, a precise symmetric manipulation of the detuning of the atoms to the system can produce the desired phase shift and set the evolution of the system on a trajectory that produces a pure 2-photon Fock state.

When this two-atom system is examined in a non-perfect cavity, it is observed that if the evolution starts with both atoms excited, the entire system decays to the ground state. If however, only one atom is excited, the cavity does not decay completely but the atoms remain in an antisymmetric entangled state. This is caused by the fact that in this type of evolution, the emission of a photon by one atom in the cavity interferes destructively with the emission of a photon by the other atom.

Then finally in section 3.5 the coherent evolution of the cavity and the emitted
photon is simulated by dividing the space in spatio-temporal modes. The simulation does not show any noticeable effect of the distance of a detector on the evolution of a cavity. Not done, but still worthwhile to examine the same system with a Mach-Zehnder interferometer between the cavity.
APPENDIX A

THE OCTAVE SCRIPTS USED

Due to size, the scripts written in Octave will not be in this document but are uploaded on Pastebin.

- Producing the eigenvalues (and eigenvectors) of the system in function of the detuning:
  http://pastebin.com/rby8A7qq
- Simulation of an arbitrary number of atoms in a cavity
  http://pastebin.com/ERTkkjQq
- The Langevin equation and Master equation:
  http://pastebin.com/LZ4pR536
- The quantum jump operator Monte Carlo method
  http://pastebin.com/KXUNGh9R
- A leaking cavity with the output photon explicitly modelled:
  http://pastebin.com/WXNcSKSu
- Simulation of the coherent evolution of the emitted photon:
  http://pastebin.com/UqgmjtfW


