EOQ DECISION MODELS IN CASE OF NON-STATIONARY PRICE EVOLUTION

Matthias Deceuninck

Supervisors: Prof. dr. Dieter Fiems, prof. dr. ir. Stijn De Vuyst
Counsellor: Eline De Cuypere

Master’s dissertation submitted in order to obtain the academic degree of
Master of Science in Industrial Engineering and Operations Research

Department of Telecommunications and Information Processing
Chairman: Prof. dr. ir. Herwig Bruneel

Department of Industrial Management
Chairman: Prof. dr. El-Houssaine Aghezzaf

Faculty of Engineering and Architecture
Academic year 2013-2014
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June 2, 2014
Matthias Deceuninck
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This dissertation is the result of the last challenge of my Master studies at the Faculty of Engineering and Architecture of Ghent University, entitled Industrial Engineering and Operations Research. I would like to thank some people who supported my research during the year.

First, I would like to acknowledge my gratitude to my first supervisor, prof. dr. ir. Stijn De Vuyst. It was a pleasure working with you! Besides good feedback, you also gave new insights and some new interesting research directions. You made me acquainted with phase-type distributions for instance. I enjoyed our meetings because you always talked with great enthusiasm about statistics and distributions.

Prof. dr. Dieter Fiems was my second supervisor and I appreciate it that he could find some time in his busy schedule to help me with my thesis. It was his idea to complement my research with an investigation of a Markov Decision Process which is stationary in the long-term but seems non-stationary in the short-term.

And finally, my family and friends for their support and advice throughout the year.

Ghent, June 2, 2014
Matthias Deceuninck
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Summary

Key words Inventory management, EOQ, geometric Brownian motion, Markov decision process
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MATTHIAS DECEUNINCK

Supervisor(s): Prof. dr. Dieter Fiems, prof dr. ir. Stijn De Vuyst, Eline De Cuypere

Abstract—Traditional inventory models assume that the purchase cost remains constant along the cycle or at least have a constant mean value. However it is widely known that prices are subject to certain market tendencies which shift the mean. The impact of non-stationary price evolutions on the optimal policy and total cost of an inventory system has been studied based on two models: Geometric Brownian Motion and a quasi-stationary Markov chain. To evaluate policies we used both simulation and Markov Decision Processes (MDP).

Keywords—Inventory management, EOQ, geometric Brownian motion, Markov decision process

I. INTRODUCTION

INVENTORY management has been an area of intense inquiry in the fields of operations management and operations research. The impact of price fluctuations is however a rather unexplored scientific domain. Product prices do not remain constant but they are subject to certain market tendencies. Technology, weather patterns, seasonal variations, political and regulatory changes all may affect the price. On top, there is also an upward tendency of the price level of goods in most countries, called inflation, which makes the time series of prices non-stationary. Two inventory systems are considered. In a first model, the geometric Brownian motion (GBM), given by equation (1), is used as a starting point to model the unit purchase cost as well as the fixed order cost.

\[ X_t = X_0 e^{(\mu - \frac{\sigma^2}{2})t + \sigma W_t} \]  

(1)

where \( \mu \) is the drift, \( \sigma \) the volatility and \( W_t \) a regular Brownian motion process.

The GBM is frequently invoked in economic analyses to model for example stock prices and natural resource prices. The second model consists of a Markov decision process in which the price evolution consists of a triangular pattern. The market moves between an upward and a downward trend. The MDP is stationary in the long-term because the probabilities and price differences remain the same. When observing a small time window however, it looks like a non-stationary price evolution, either going up or down.

II. MODEL WITH GBM PRICE EVOLUTION

As mentioned, it is assumed that the unit purchase cost and the fixed order cost are modeled by two independent geometric Brownian motions. To extend the model with seasonality and sudden price jumps, the modified GBM model of Ladde and Wu [1] is used. A year is partitioned into four quarters with each quarter having its own drift.

In contrast to Ladde and Wu, the jumps do not have to occur at the end of a quarter but can happen at any time. Fig.1 depicts sample paths of the modified GBM. Note that the blue and red curve are almost consistently higher than the expected values because some price jumps have occurred while jumps are not considered for the green curve.

![Example sample paths of modified GBM](image)

Furthermore, it is assumed that:
- the daily demand is gamma distributed,
- the delivery lead time is gamma distributed,
- a backlog cost is charged for any demand that cannot be met instantaneously,
- there are no capacity constraints and only one product is involved,
- the fixed order cost is independent of the order quantity.

III. RESULTS

Including inflation or time-value of money turns out to be most critical for low demand products with a high order cost compared to the inventory carrying rate. Adjusting the EOQ for the quarterly inflation (2) is also optimal under the given conditions.

\[ Q^* = \frac{EOQ}{\sqrt{1 - \frac{\mu_q}{h}}} \]  

(2)

with \( h \) the annual inventory carrying charge rate.

The reason for this, besides that GBM is also exponential in the mean, is that the GBM has no incentive to move to its expected value in the short term. The Markov property...
and the constant expected return ensure relative lognormal increments independent of the price. In other words, the probability of a price increase of 5% when the price is low is equal to the case when the price is high.

But under some conditions it may be better to place orders when the inventory is not empty but lower than a certain threshold. A policy is proposed where an order may be placed at the end of a quarter or period when the following period is predicted to be subject to a much higher inflation. In the example of fig.1, this is for instance the case at the beginning of $q_1$ and $q_3$. It turns out that the policy leads to lower costs when the interest rate $\alpha$ is lower than the average inflation rate. In the opposite case, costs should be delayed as much as possible because of the time-value of money.

**IV. Model with Markov Decision Process**

An inventory system of a product is considered that is subject to demand as well as price fluctuations. The inter-arrival time between two consecutive demands is assumed to be negative binomially distributed. This is depicted in fig.2. Furthermore, the inventory considers immediate delivery and order decisions are made at discrete time slots.

Concerning the price fluctuations, the market can be either subject to an upward or a downward trend, respectively denoted as a bull market and bear market. For the former, the probability of a price increase is bigger than the probability of a price decrease while the opposite is true for the latter. The transition probabilities between these two market trends are chosen in such a way that it is more likely to have a transition from the bull market state to the bear market state when the price is high. Both analytical considerations and simulations were used to find good parameter values.

**V. Results**

In our first example the inventory system has no capacity constraint and the price increases linearly with an interest rate $i$ of 10%: $P_{k+1} = P_k(1 + i)$ and $P_1$ equal to 10 dollars. The holding cost equals 0.012 dollar per unit per day and the fixed ordering cost equals 90 dollar. This leads to an EOQ of 39. Four general observations can be made when analyzing the optimal order policy:

- the order quantity is higher than the EOQ at market bottoms for both market conditions. Thus the optimal order quantity is high when the price is low.
- the order quantity is lower than the EOQ at market tops for both market conditions.
- in between, the order quantity is higher and lower than the EOQ for respectively, a bull market and bear market.
- the order quantity decreases with increasing price.

When observing the impact of the capacity constraint, it can be noted that the order quantity raises for some cases in a bear market. For some price levels, a higher quantity is ordered which is closer to the economic order quantity. This can be explained by the fact that the system will speculate less on a price decrease, because it will not be able to fully benefit from it.

**VI. Conclusions**

The geometric Brownian motion is not really capable to capture all properties of the price evolution of a product. The classical stock replenishment systems are quite robust against this stochastic price evolution when looking to the optimal order policy.

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<tr>
<td>$Q$</td>
<td>products</td>
<td>Order quantity</td>
</tr>
<tr>
<td>$R$</td>
<td>products</td>
<td>Reorder point</td>
</tr>
<tr>
<td>$TC$</td>
<td>dollar</td>
<td>Total cost</td>
</tr>
<tr>
<td>$K$</td>
<td>dollar</td>
<td>Fixed order cost</td>
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<tr>
<td>$D$</td>
<td>products per year</td>
<td>Annual demand</td>
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<tr>
<td>$C_h$</td>
<td>dollar per product per year</td>
<td>Holding cost</td>
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<tr>
<td>$P$</td>
<td>dollar</td>
<td>Unit purchase cost</td>
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## EOQ models adjusted for inflation

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<th>Description</th>
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<tr>
<td>$P_t$</td>
<td>dollar</td>
<td>Unit purchase cost at time instant $t$</td>
</tr>
<tr>
<td>$K_t$</td>
<td>dollar</td>
<td>Fixed order cost at time instant $t$</td>
</tr>
<tr>
<td>$h$</td>
<td>%</td>
<td>Annual carrying charge rate (as percentage of price)</td>
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<tr>
<td>$IL_t$</td>
<td>products</td>
<td>Number of units in inventory at time instant $t$</td>
</tr>
<tr>
<td>$i$</td>
<td>%</td>
<td>Inflation rate, denoted as % per year</td>
</tr>
<tr>
<td>$T$</td>
<td>years</td>
<td>Replenishment cycle length</td>
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<tr>
<td>$Q^*$</td>
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### Geometric Brownian motion

#### model 1

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<tr>
<td>$\mu$</td>
<td>%</td>
<td>Drift rate of Geometric Brownian Motion</td>
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<tr>
<td>$\sigma$</td>
<td>%</td>
<td>Volatility of Geometric Brownian Motion</td>
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<tr>
<td>$\alpha$</td>
<td>%</td>
<td>Interest rate</td>
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#### model 2

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<th>Symbol</th>
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<tr>
<td>$q_i$</td>
<td>–</td>
<td>quartile $i$</td>
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<tr>
<td>$\zeta$</td>
<td>%</td>
<td>Probability of a price jump in a year</td>
</tr>
<tr>
<td>$\beta$</td>
<td>%</td>
<td>Parameter for height price jump</td>
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#### model 3

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<th>Symbol</th>
<th>Units</th>
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<tr>
<td>$L$</td>
<td>days</td>
<td>Replenishment lead time</td>
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<tr>
<td>$F_B$</td>
<td>dollar</td>
<td>Penalty cost for backlogging</td>
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<td>$IP$</td>
<td>products</td>
<td>Inventory position</td>
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<tr>
<td>$\kappa_D$</td>
<td>–</td>
<td>Shape factor of gamma distributed demand</td>
</tr>
<tr>
<td>$\theta_D$</td>
<td>–</td>
<td>Scale factor of gamma distributed demand</td>
</tr>
<tr>
<td>$\kappa_{LT}$</td>
<td>–</td>
<td>Shape factor of gamma distributed delivery time</td>
</tr>
<tr>
<td>$\theta_{LT}$</td>
<td>–</td>
<td>Scale factor of gamma distributed delivery time</td>
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### Markov Decision Process

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<tr>
<td>$p$</td>
<td>%</td>
<td>Probability of moving to next demand phase (parameter of negative binomial distribution)</td>
</tr>
<tr>
<td>$r$</td>
<td>#</td>
<td>Number of demand phases (parameter of negative binomial distribution)</td>
</tr>
<tr>
<td>$C$</td>
<td>products</td>
<td>Capacity of inventory storage</td>
</tr>
<tr>
<td>$\alpha$</td>
<td>%</td>
<td>Probability of price increase in bull market</td>
</tr>
<tr>
<td>Symbol</td>
<td>Units</td>
<td>Description</td>
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<tr>
<td>--------</td>
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<tr>
<td>$\beta$</td>
<td>%</td>
<td>Probability of price decrease in bull market</td>
</tr>
<tr>
<td>$\gamma$</td>
<td>%</td>
<td>Probability of price increase in bear market</td>
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<tr>
<td>$\lambda$</td>
<td>%</td>
<td>Probability of price decrease in bear market</td>
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<tr>
<td>$\epsilon$</td>
<td>%</td>
<td>Probability of transition bull market to bear market</td>
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<tr>
<td>$\delta$</td>
<td>%</td>
<td>Probability of transition bull market to bear market</td>
</tr>
<tr>
<td>$v$</td>
<td>%</td>
<td>Factor used to model price transitions</td>
</tr>
<tr>
<td>$T_{avg}$</td>
<td>time units</td>
<td>Average time of a bull or bear market cycle</td>
</tr>
<tr>
<td>$C_h$</td>
<td>dollar per product per year</td>
<td>Yearly holding cost per product</td>
</tr>
<tr>
<td>$c_h$</td>
<td>dollar per product per day</td>
<td>Daily holding cost per product</td>
</tr>
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</table>
Chapter 1

Introduction

1.1 Problem statement

The problem of this master thesis is situated in the area of inventory management. It is an important business field which accounts for the success or failure of modern businesses in all kinds of industries. Inventory ties up working capital and incurs holding costs, reducing profit every day excess stock is held. The holding cost includes housing costs, material handling cost, insurance, obsolescence, interest, etc. and can be estimated as 20-45% of the inventory value. Hence if the company does not want to lose its competitive edge, investment in inventory should neither be excessive nor insufficient. Every excess in inventory is a waste and should be identified and eliminated. Insufficient inventory, on the other hand, creates problems such as production interruptions and stock-outs, which lead to poor service levels and unsatisfied customers.

Inventory management is necessary at different locations within an organization. The scope of inventory management is broad and includes for example, besides replenishment and carrying costs of inventory, the control of lead times and spares risk assessment. Another activity of an inventory manager consists of forecasting the future cost of goods. By estimating reasonable inflation and interest rates, better decisions can be made towards order quantities.

Classic inventory models order a batch quantity of size $Q$, when the inventory position declines to or below the reorder point $R$. This is denoted as an (R,Q) policy and order reviews can be either continuous or periodic. In the title of the master thesis, the acronym "EOQ" is included, which refers to the model of Ford W. Harris. He was the first to publish a general, mathematical solution for optimal order quantities [11]. His EOQ model was however subject to a relatively restrictive set of assumptions
and he knew that each case calls for a trained judgement. He stated in his three-page article:

"The writer does not wish to be understood as claiming that any mere mathematical formula should be depended upon entirely for determining the amount of stock that should be carried or put through an order."

Nonetheless, the EOQ model has had a huge impact in the fields of operations management and operations research. Determining the optimal stock level started to become a science. It was not longer an art. The model is the foundation for literally thousands of later studies. It also remains relevant because many companies still use it in practice due to its simplicity and robustness with respect to the estimation of its parameters (demand, fixed order cost, holding cost). In section 1.2 the original EOQ model is briefly explained.

Determining the optimal reorder point $R$ and quantity $Q$ on the basis of the classic EOQ model is not longer optimal in a more stochastic and dynamic environment. Relaxing some assumptions and adjusting the decisions to the economic situation can lead to significant cost savings despite the robustness. Traditional inventory models assume for instance that the purchase price is constant along the cycle. However, commodity prices and other costs have shown to be stochastic in practice. Considerable price fluctuations over short periods prevail in today’s unstable global economic climate. New technology, changes in market conditions, variations in supply and demand as well as speculation may all arise such fluctuations. A constant price is only realistic in the existence of contracts with predetermined prices. With continuous changing costs as material costs, housing costs, etc. ordering at the right time can result in a lower total cost and thus increasing the profit margins of the firm.

### 1.2 The original Economic Order Quantity model

The EOQ can be derived from the equation of total cost (1.1). To obtain the optimal order quantity you take the derivative and set it equal to zero. Under the given assumptions, which are partly listed below, the optimal order quantity can be determined by using the only three relevant components: holding cost, ordering cost and demand. This classic EOQ formula is given in equation 1.2

\[
TC = K \frac{D}{H} + C_b \frac{Q}{2} + PD
\]  

(1.1)
1.2. THE ORIGINAL ECONOMIC ORDER QUANTITY MODEL

\[ Q^* = \sqrt{\frac{2 K D}{C_h}} \]  

(1.2)

**Figure 1.1:** Total cost curve and determining Economic Order Quantity

The EOQ model simplifies strongly the reality, and is rarely optimal for companies. The following assumptions of the model are not always realistic:

1. The replenishment is made instantaneously, the whole batch is delivered at once.
2. The purchase price of the item is constant and there is no discount available.
3. Annual demand for the item, \( D \), is deterministic and occurs at a constant rate over time.
4. The lead time is known and constant.
5. Only one product is involved.
6. Holding cost per unit \( C_h \) is constant and independent of number of units.
7. Ordering cost \( K \) is constant and independent of size.
8. No obsolescence. The risk of damaged or obsolete goods is not considered in the model.
1.3 Research goals

Like many other studies, we want to extend the traditional EOQ model by relaxing some assumptions. In this dissertation the impact of non-stationary price evolution on the optimal policy and the total inventory cost will be analyzed. Product prices have indeed shown not to remain constant as assumed by the traditional EOQ model. They are subject to certain market tendencies and fluctuate throughout history and this may affect the optimal order policy. EOQ is just one of the possible order strategies, and optimizing its parameters will not necessarily mean that you optimize the whole order process. In a more stochastic and dynamic environment for which the simplifications of the EOQ model do not longer hold, different strategies may prove to be more optimal. The (R,Q) policy where an order of fixed quantity Q is placed, is for instance substituted by a policy which accounts for the current economic state. Both simulation and Markov decision processes are used to analyze different models. In chapter 3, the price is subject to constant inflation. The mean price and its variance increase over time and this may lead to different order quantities. The geometric Brownian motion$^1$ is used as starting point to model the price evolution in chapter 4. In an extended model, we include also seasonal behavior. Finally, chapter 5 considers a price evolution which fluctuates between an upward and downward trend. The evolution seems non-stationary in the short-term because of the trend, but it is stationary if we look at the long-term behavior.

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$^1$GBM: a continuous-time stochastic process in which the logarithm of the randomly varying quantity follows a Brownian motion with drift.
Chapter 2

Literature review

Inventory modeling has been an area of intense inquiry in operations management and operations research. Starting from the simple deterministic EOQ model, the field of operations management has developed much more advanced inventory models that incorporate stochastic and correlated demands, multiple products and multiple echelons of inventory. In the literature, many studies relax some assumptions and make the EOQ model more applicable for a specific case.

The Economic Production Quantity (EPQ) relaxes for example the assumption that the replenishment is made instantaneously, while other extensions consider traditional discount situations (such as "all units" and "incremental" discounts) or stochastic lead times. The latter is the case where constant and known lead times do not apply in practice. The assumption is for instance less applicable when suppliers are located far away from the company and shipments spend a lot of time in transit and are subject to high uncertainty.

An important assumption of the model is the constant rate demand. Extensions of the classic EOQ model have studied extensively the impact of demand fluctuations. For a linearly increasing demand pattern, Resh et al. and Donaldson established an algorithm to determine both the optimal order quantity and timing [21] [6]. Barbosa and Friedman then generalized the solutions for various, similar demand models [2]. Real-life situations as deteriorating products and shortages are included by Dave [7], following the approach of Donaldson.

An interesting study was done by Song and Zipkin, who determined optimal inventory policies in a fluctuating demand environment with a Markov process [23]. The demand process is not considered purely random. It assumes some identifiable factors or quantities which determine the demand envi-
ronment (e.g. product sensitivity to economic conditions: demand for many products respond in part to changes in certain basic economic variables). In this way, the ‘world’ affects demand by its state (Markov-modulated Poisson process). The lead times are assumed to be phase-type distributed.

In contrast to fluctuations in demand, the impact of price fluctuations is a rather unexplored scientific domain. Kalymon studies a multi-period stochastic inventory model, in which the procurement price is driven by a Markovian stochastic process [16]. The distribution of demand in each period depends on the current cost. J.-T. Teng et. al have established an algorithm assuming that not only the demand function but also the purchase cost is fluctuating with time [25]. In a numerical example, it is shown that the total cost can be substantially reduced. However, this algorithm relates to products with a continuously decreasing price, such as high-technology products, and no lead time is assumed for delivery. To generalize this, De Cuypere et al. proposed a discrete-time Markovian model with stochastic lead times that accounts for both demand and market price fluctuations [9]. They assume a geometrical distributed lead time and a demand following a Bernoulli distribution.
Chapter 3

EOQ model adjusted for inflation

In this chapter the results and analysis are presented in the case of constant inflation rates. In order to gain first insights in the dynamics of non-stationary markets, some different models of frequently faced situations are considered in this section. It is also determined how much the total cost can be reduced under certain conditions.

3.1 Introduction

History has shown that prices most likely do not remain constant. Commodity prices are subject to certain market tendencies. When the price level of goods and services tend to persistently increase over a period of time, it is called inflation and is usually measured in terms of a specific annual percentage. Deflation is the opposite case and refers to a general decline in prices. Deflation is rare and occurs only during recessions. National inflation rates vary widely in individual cases. The inflation rate in Belgium averaged for instance 5.90 percent from 1921 until 2014. While the inflation rates have declined for most countries for the last several years, in some countries it is still in the range of 10% to 56% (Venezuela) [26].

Inflation has many different possible causes. The main Keynesian theories [3] consider demand-pull, cost-push, and built-in inflation. In the case of demand-pull, inflation is caused by aggregate demand being more than the available supply. A common cause of demand-pull situations is an increase in consumer spending because of increased optimism caused by a boom in the economy. When people are more confident about their financial future, they tend to spend more, contributing to a rise in prices. Decreasing taxes can also lead to inflation, because people have more money to spend. Cost-push inflation occurs when manufacturers and businesses raise prices as a result of shortages, or as
a measure to balance other increases in production costs. Built-in inflation happens as a result of previous increases in prices caused by demand-push or cost-pull. In this type of situation, people expect prices to continue to rise, so they push for higher wages. This raises costs for manufacturers, which then raise the cost of goods to compensate, causing a cycle of inflation. There are also many things that can cause short-term increases in prices, including natural disasters, wars and decreases in natural commodities.

Although the general inflation may be currently low, the cost of several raw materials that companies face are going up considerably faster than the CPI. In the article it has been indicated that these costs are being offset by below-normal increases in wages in the CPI. Different indicators, for example the Reuters CRB index may give different signals of inflation. It can be concluded that there is no absolute and objective gauge of inflation and almost every company will have products that are subject to high inflation at some point. The investigation of inflation is therefore still worthwhile. The impact of inflation on the economic order quantity will be discussed in section.

3.2 EOQ models adjusted for Inflation

Classic economic order quantity formulas assume however constant prices and do not take inflation into account. In this section, the impact of inflation while determining the replenishment order sizes, will be discussed. The existing literature investigated inflationary conditions as well as the concept of time-value of money. These two concepts may sometimes lead to confusion and are therefore explained with an example below.

---

2Consumer Price Index (CPI): measures changes in the price level of a market basket of consumer goods and services purchased by households.
3Reuters CRB index: a commodity futures price index, including commodities aluminum, coffee, copper, crude oil, etc.
3.2. EOQ MODELS ADJUSTED FOR INFLATION

Concepts time-value of money and inflation

Let us examine the costs associated with an inventory system for several years. The total expenses generated by the inventory manager during the entire period include three elements:

1. the total purchase cost, which is determined by \( \sum_{j=0}^{n} Q_j P_j \) where \( Q_j \) is the order size of the j-th order and \( P_j \) the unit purchase cost at that time instant.

2. the total fixed order cost \( \sum_{j=0}^{k} K_j \) where \( K_j \) is the fixed order cost at time \( j \).

3. the holding cost \( h \int_{0}^{T} IL_t P_t \, dt \) where \( IL_t \) is the number of units in inventory at time instant \( t \).

These three elements may all be subject to inflation. Increments in wages, costs of material, utilities, etc. may lead for instance to a \( P_n \) that is much higher than \( P_0 \). Now, the sum of the three gives the total cost which we will pay during the entire period and which incorporates the inflationary factor.

So far we have neglected the time-value of money in our calculations. Now suppose the following situation where we want to open a bank account for the inventory manager today and store the equivalent value needed to cover all expenses during the period. How much money do we need to deposit? This sum will be smaller than the sum of the above three elements, because the inventory manager will earn interest on the deposit before the expenses are made.

Thus money does not have a time-value because of inflation. But inflation should be taken into consideration when determining capital decisions.

Aggarwal [24] specified a situation where the optimum size of the next reorder quantity \( Q^* \) is calculated, considering that any future reorder will be subject to a constant rate of inflation. Savings from purchasing a larger lot now instead of purchasing later at an inflated price are traded off with the extra holding cost holding minus the proportional decrease in reordering cost. This trade-off is shown by equation [3.1]

\[
\left( Q^* - Q_0 \right) P_t i T_0 = \left( Q^* - Q_0 \right) P_t T_0 h + \frac{1}{2} \left( Q^* - Q_0 \right) P_t h (T^* - T_0) - \left( \frac{Q^* - Q_0}{Q_0} \right) K \tag{3.1}
\]

\( Q^* - Q_0 \) savings purchase cost
\( P_t i T_0 \) extra inventory cost
\( \frac{1}{2} \left( Q^* - Q_0 \right) P_t h (T^* - T_0) \) fewer reorders
3.2. EOQ MODELS ADJUSTED FOR INFLATION

It is assumed that the unit purchase cost $P_t$ is the only variable that is directly subject to linear inflation, denoted as $i \%$ per year. At time $t$, the unit purchase cost is inflated to $P_t = P_0 (1 + i t)$. The annual carrying charge rate $h$ and the reordering cost $K$ remain the same, irrespective of inflation on purchase cost. $T^*$ and $T_0$ denote respectively the time that is covered by ordering $Q^*$ and $Q_0$ units.

In the paper it is shown that in case of a constant inflation rate, $i$, an items optimal reorder size $Q^*$, remains equal to the classical EOQ for low values of $i$. Above a certain critical value of $i$, $Q^*$ starts increasing with increasing $i$.

To extend the research of Aggarwal, we investigated the case where the total purchase cost (including the fixed order cost) is subject to exponential inflation. We compared the optimal total costs $TC^*$ with those obtained by applying the standard EOQ model. The same inflation rate is assumed for the fixed cost $K_t$ and the unit purchase cost $P_t$. They both follow an exponential function: $X_t = X_0 e^{(i t)}$.

Now, because they both follow the same exponential function, all the future replenishments are of equal size and the cost per cycle increases by the same function. This allows us to get a tractable analytical result.

Figures 3.1 and 3.2 give the results for different parameters. The graphs show that the standard EOQ model is very robust for high demand products which have a low fixed cost compared to the holding cost. The replenishment cycle time $T$ is low under these conditions and inflation cannot have a big impact. That is for instance the reason why the EOQ model has been effectively employed in retail sectors of the economy for many years. The percentage cost reductions with respect to the standard EOQ model are calculated as:

$$\text{reduction\%} = \frac{TC_{EOQ} - TC^*}{TC_{EOQ}}$$

Only small reductions in total cost are obtained by applying equation 3.3. These small percentages can lead to significant cost reductions however.

$$Q^* = \frac{EOQ}{\sqrt{1 - \frac{i}{h}}}$$

---

4 Annual carrying charge rate $h$: percentage which expresses how the stock cost relates to the unit purchase cost. It is a different way of expressing the inventory cost. The holding cost per unit per year $C_h$ is for instance equal to $h P_t$. 
3.2. EOQ MODELS ADJUSTED FOR INFLATION

Figure 3.1: Performance of the inflation model compared to the standard EOQ model for different fixed cost/holding cost ratios. The yearly demand is equal to 500 units, $h$ equal to 30% and $P_0$ equal to 10 dollars.

Figure 3.2: Performance of the inflation model compared to the standard EOQ model for different yearly demand. The initial fixed cost $K_0$ is set equal to 300 dollars per reorder, $h$ equal to 30% and $P_0$ equal to 10 dollars.
[20] extends the previous research and incorporates the effects of inflation and time-value of money in an EOQ model with a random product life cycle. The inflation is also considered exponential and the reordering cost is also subject to it. An algorithm is proposed in the paper, which starts from equation 3.4.

\[ T = \sqrt{\frac{2K}{PD(h + \lambda - i)}} \]  

(3.4)

This is the optimal cycle length for the basic EOQ modified for inflation and with the carrying charge rate approximately adjusted for the risk of obsolescence. This value is then increased or decreased dependent on the parameters. The optimal cycle length, \( T^* \), increases (decreases) for decreasing (increasing) ratio \( \frac{PD}{K} \). Including time-value of money or inflation turns out to be most critical in situations with high fixed costs and purchase prices, while demand and inventory carrying rates are low. In these cases, the highest savings can be made.

It should be noted that the unit cost \( P \) does not affect the optimal order quantity in the infinite planning horizon EOQ model. Consequently, a finite horizon inventory model (accounting for the time-value of money) turns out to be theoretically superior. It is also of greater practical utility, because an infinite planning horizon does not exist in real life. These insights are taken into account in the next chapters.
Chapter 4

Geometric Brownian motion

4.1 Introduction

There are a lot of factors that can affect commodity prices including seasonal variations, weather patterns and market conditions. Considering a constant inflation does not capture all these short-term causes of price fluctuations. In this chapter, the geometric Brownian motion (GBM) is used as starting point to model the unit purchase cost as well as the fixed order cost. This is the same assumption that many recent engineering economic analyses have relied on. The GBM is frequently invoked as a model for such diverse quantities as stock prices, natural resource prices, and the growth in demand for product or services. Black and Scholes use this model of stock price behavior for pricing European call and put options, as well as their variations for a few of the more complex derivatives. This BS model is still the most popular model of financial market. Other applications of the GBM and the validity of its assumptions are discussed in [19]. In the paper it is shown that caution should be taken before assuming that a particular data set follows the GBM process. They concluded that for some data sets, the GBM process may be appropriate, based on the criteria of normality and independence. For other data sets however, some of the assumptions (constant drift and volatility for instance) may not be appropriate.

4.2 Why geometric Brownian motion

The geometric Brownian motion has some relevant properties for price behavior. In this section some of these properties will be discussed. The formula of the GBM is given by equation (4.1).

\[ X_t = X_0 e^{(\mu - \frac{\sigma^2}{2})t + \sigma W_t} = X_0 e^{F(t)} \]
4.2. WHY GEOMETRIC BROWNIAN MOTION

Where $X_0 > 0$ is the initial value and $W_t$ is a Wiener process (see below). Drift $\mu$ and volatility $\sigma$ are constants.

The drift $\mu$ is used to model deterministic trends, while the volatility $\sigma$ models a set of unpredictable events occurring during the motion. Note that GBM can not take on negative values. The expected value and the variance of the process are easily derived:

$$E(X_t) = X_0 e^{\mu t}$$  \hspace{1cm} (4.2)

$$\text{Var}(X_t) = X_0^2 e^{2\mu t} (e^{\sigma^2 t} - 1)$$  \hspace{1cm} (4.3)

A sample path of the GBM is given in figure 4.1.

![Sample path of geometric Brownian motion](image)

**Figure 4.1:** Example of sample paths of geometric Brownian motion with $\mu$ equal to 0.05 and $\sigma = 0.15$ (blue) or $\sigma = 0.05$ (red). The green curve shows the expected value in function of $t$.

The green curve gives the expected value of the price at each time instant $t$ and is equal to $X_0 e^{\mu t}$. The exponential curve is clearly observable and the absolute price differences increase. This corresponds to a yearly inflation of $e^{\mu} - 1$. With $\mu$ equal to 0.05, the price increases 5.13\% every year for instance.

**Wiener Process or Brownian motion**

A Wiener process is a continuous-time stochastic process. It is often called the standard Brownian motion (BM) because in physics it is used to describe this transport phenomenon of particles suspended in a fluid. It is formally expressed by:

$$\Delta x = \epsilon \sqrt{\Delta t}$$  \hspace{1cm} (4.4)
And is characterized by the following three properties.

1. $\epsilon$ follows a standardized normal distribution $N(0, 1)$

2. $W_t$ is continuous with $W_0 = 0$

3. $W_t$ has independent increments $\Delta x$ for any two different intervals of length $\Delta t$ and which are normally distributed. $\Delta x$ has also a normal distribution with mean 0 and variance $\Delta t$.

A generalized Wiener process is the basic Wiener process scaled by a factor $b$ and with a drift $a$. That is,

$$\Delta x = a \Delta t + b \epsilon \sqrt{\Delta t}$$

$F(t)$ in equation 4.1 is such a generalized Wiener process with scale factor $\sigma$ and a drift of $\mu - \frac{\sigma^2}{2}$. Figure 4.2 depicts this graphically.

![Sample path of a Wiener process and a generalized Wiener process with $a = 0.3$ and $b = 1.5$.](image)

**Figure 4.2:** Sample path of a Wiener process and a generalized Wiener process with $a = 0.3$ and $b = 1.5$.

**Markov property**

The first property we discuss is called the Markov property. It states that the present value of a variable is the only relevant piece of information for predicting the future.
4.2. WHY GEOMETRIC BROWNIAN MOTION

\[ X_{t+h} = X_0 \, e^{(\mu - \frac{\sigma^2}{2}) \, (t+h) + \sigma W_{t+h}} \]

\[ = X_t \, e^{(\mu - \frac{\sigma^2}{2}) \, h + \sigma (W_{t+h} - W_t)} \]

\[ = X_t \, e^{(\mu - \frac{\sigma^2}{2}) \, h + \sigma \sqrt{h}} \]

where we used the above property 3 that increments of \( W_t \) are independent and normally distributed.

Therefore, for any \( 0 \leq s < t \) and \( h > 0 \), if \( X_t \) is known, \( X_{t+h} \) is independent of \( X_s \), which shows that the GBM \( X_t \) is a Markovian process.

The past history of the variable and the way that the present has emerged from the past are irrelevant. The Markovian property is consistent with the weak form of market efficiency. This states that the present price of a stock impounds all the information contained in a record of past prices. If the weak form of market efficiency were not true, technical analysts could make above-average returns by interpreting charts of the past history of stock prices. There is very little evidence that they are in fact able to do this. It is competition in the marketplace that tends to ensure the weak-form market efficiency holds (\([4]-\text{chapter 8}, [15]\)). There are many investors watching the stock market closely. The reason why this is an acceptable assumption for stock prices is explained by the following example.

Markov property of stock prices

Suppose that it was discovered that a particular pattern in stock prices predicts with great confidence a subsequent steep price rise. Investors would attempt to buy a stock as soon as the pattern was observed to cash in on the forthcoming increase in stock price. Stock owners however would not be willing to sell and this would lead to an immediate jump in the stock price. The observed effect would be eliminated, as would any profitable trading opportunities. Thus trying to make a profit from the pattern leads to a situation where a stock price, at any given time, reflects the information in past prices. A forecast about favorable future performance leads instead to favorable current performance.
4.2. **WHY GEOMETRIC BROWNIAN MOTION**

**Constant expected return**

A constant expected return is another key aspect of stock prices. The expected percentage return required by investors is the same for a stock price of $10 as for a stock price of $50, and thus independent of the stock price. Investors require the same expected percentage per annum. Equation 4.2 shows that, when the variance $\sigma^2$ is zero, the stock price grows at a continuously compounded rate of $\mu$ per unit of time. In practice a stock price does exhibit volatility and a reasonable assumption therefore is that the variability of the percentage return in a short period of time, $\Delta t$, is the same regardless of the stock price. In other words, an investor is just as uncertain of the percentage return when the stock price is $50 as when it is $10.

**Independence and normality of increments**

The geometric Brownian motion assumes that the percentage changes over equal length, non-overlapping intervals are independent and identically distributed. For each $t$, the ratios $L_i = \frac{X(t_i)}{X(t_{i-1})}$ are independent lognormal random variables which reflects the fact that it is the percentage of changes of the stock price that are independent, not the actual changes $X(t_i) - X(t_{i-1})$. This is an important difference with the standard BM, where the actual increments are independent.

$$L_1 = \frac{X(t_1)}{X(t_0)} = e^{(\mu - \frac{\sigma^2}{2})t_1 + \sigma W_{t_1}} = e^{F(t_1)} \quad (4.5)$$

$$L_2 = \frac{X(t_2)}{X(t_1)} = e^{F(t_2) - F(t_1)} \quad (4.6)$$

$F(t_1)$ en $F(t_2) - F(t_1)$ are independent and normally distributed and therefore we can re-write $X(t)$ as an independent product of $n$ lognormal random variables:

$$X_t = X_0 L_1 L_2 ... L_n \quad (4.7)$$

**Shortcomings of the model**

At some points the GBM is not completely realistic. We list some of the assumptions below.

1. GBM assumes constant volatility. In reality however, volatility changes over time.

2. In real stock prices, returns are usually not normally distributed. They have higher kurtosis and a negative skewness.
3. GBM does not include cyclical or seasonal effects. It cannot really capture periods of constant values for instance.

4. The constant expected return may not be realistic for commodity goods. When the price is low, there may be a higher probability of a price increase than the case wherein the price is already high.

### 4.3 Assumptions of model

One way of gaining an intuitive understanding of this stochastic process is to simulate the behavior of the GBM price process. This involves dividing a time interval into many small time steps and randomly sampling possible paths for the variable. In this section Monte Carlo simulations will be carried out to study the differences in total cost. This will be done for the traditional EOQ model and the EOQ model with inflation under the assumption that the prices follow the geometric Brownian motion, as described above. It will also be studied if the policy can be optimized.

In a first model, a few assumptions are made in order to simplify reality and gain first insights in the model. These assumptions are listed below.

1. Delivery of the products is instantaneous. Delivery lead time $L$ is in other words equal to zero.

2. The inventory system is subject to price fluctuations. The unit purchase cost $P_t$ and the fixed ordering cost $K_t$ both follow a geometric Brownian motion with the same characteristics but independent of each other.

3. Annual demand for the item, $D$, is deterministic and the demand occurs at a constant rate over time.

4. Orders are only made when the inventory level is zero.

5. Only one product is involved and there are no capacity constraints.

6. Fixed order cost $K$ is independent of order quantity $Q$.

Because of the assumptions that the delivery lead time and reorder point are zero and the demand is constant, we can derive for each time instant $t$ when the next order will be made. An order of $Q$
products means that the next order takes place at a time $Q/D$ after the order moment (Fig. 4.3). When ignoring time-value of money, the holding cost for the replenishment cycle of order $j$ can be measured as:

$$Holding\ cost\ cycle\ j = \frac{hP_j Q_j^2}{2D}$$

Here, we use the unit purchase price $P_j$ at the moment of the order to measure the holding cost for the whole replenishment cycle. Thus the inventory is reported at the amount paid to obtain it and not at its current value $P_t$.

![Figure 4.3: Inventory level in function of time with deterministic and constant demand.](image)

Monte-Carlo simulations are used with relatively short time windows. Simulating over long periods may lead to wrong decisions, because there may be too much focus on the long-term behavior. The exponential and non-stationarity property of the geometric Brownian motion (eq. 4.1) indeed implies that the prices and price differences are much higher at the end of the simulation. A simulation time of five years seems reasonable to investigate.

A difficulty that has to be encountered is the evaluation of two different inventory levels at the end of the simulation. In contrast to a real inventory system, the simulation cannot be continuous and has to be ended at one moment to make a comparison possible. If policy $\pi_1$ ends with $Q_1$ products in stock and $\pi_2$ ends with $Q_2$ products, comparing the total costs up to time $t$ leads to erroneous conclusions. The highest stock has a lower expected cost in the near future, because replenishment can be postponed. Comparing the total cost at empty stock and taking the average cost over the number of replenishments
is not a possibility because of the non-stationarity property. Different evaluation methods are possible. In a first method, we subtract the following costs from the total cost:

\[ P_t \Delta Q + \frac{h P_t (\Delta Q)^2}{2D} + \frac{\Delta Q}{Q_t} K_t \]  \hspace{1cm} (4.8)

The price \( P_t \) at the end of the simulation, in our case five years, is observed and all products left in the stock are valued at this price. \( \Delta Q \) is equal to \( D \Delta T \), with \( T \) the time between the last replenishment and the simulation time. Furthermore, the cost is reduced with the future inventory cost of the remaining stock and with a fraction of the fixed order cost \( K_t \). The fraction is calculated as \( \Delta Q \) over the hypothetical order quantity \( Q_t \) at time \( t \), if the order would be placed.

Another way of evaluating the remaining stock levels consists of randomly choosing the simulation time. The simulation time is for instance randomly picked from a normal distribution with a mean of five years and standard deviation of 0.1 years. This evaluation method benefits order policies with lower order quantities because they have a higher chance of ending with a low inventory level. The order quantities that give the lowest cost with this evaluation method are 15% to 20% lower than the actual optimal order quantities.

To evaluate the results, a reference is required. For this purpose, the traditional EOQ model is used again, wherein at each decision point the prevailing price of the fixed cost \( K_t \) is taken into account. So its decision is independent of the prices in the past or future. One orders as if the market were stationary with constant price levels and calculates the optimal quantity by equation 1.2. Order quantities will change over time, dependent on the evolution of the fixed cost.

### 4.4 Model with GBM price evolution

#### 4.4.1 Model

When determining the optimal order quantity, the inventory manager has to take three things into account. The ordering cost, the holding cost and the difference between the current and future product price should all be estimated. The actual price is of less importance than the price difference at the ordering points. If the price of the product is low at the moment, you would intuitively order more products. This is however only justified if the price does increase a lot in the near future. When the
price remains low, or increases only a small amount, the extra holding cost may outweigh the savings on the price. This is an important insight if we look at this geometric Brownian motion model. In section 4.4.3, both the optimal order quantity and the robustness of the model will be examined. But first we will take a look at the robustness of the EOQ model.

### 4.4.2 Robustness EOQ

In the real world, it is often difficult to estimate the model parameters accurately. The cost and demand parameter values used in models are at best an approximation to their actual values. Clearly, if approximated parameters are used, the realized cost will be greater than the cost of the true optimal policy. But the classical EOQ model is rather insensitive to these parameter values. A demand forecast error of +20% for instance, increases the total inventory and ordering cost only with 0.4%. More generally, the total cost increase can be expressed as a function of the demand percentage error \( x = \frac{D_{\text{forecast}} - D_{\text{actual}}}{D_{\text{actual}}} \). With a wrong forecast, the EOQ will be calculated as:

\[
Q = \sqrt{\frac{2(1 + x)DK}{H}} = \sqrt{1 + x} Q^* \tag{4.9}
\]

\[
TC^* = \frac{HQ}{2} + \frac{DK}{Q}
\]

\[
TC = \frac{\sqrt{1 + x} Q H}{2} + \frac{DK}{\sqrt{1 + x} Q} \tag{4.10}
\]

\[
= \sqrt{1 + x} \left(1 - \frac{1}{2} \frac{1 + x}{1 + x}\right) TC^*
\]

The relative cost \( \frac{TC - TC^*}{TC^*} \) in function of forecast error \( x \) is given in figure 4.4. The same robustness applies with respect to the fixed cost and inventory cost.
4.4.3 Optimal order policy

Inflation model without variance

The adjusted EOQ model for inflationary conditions is used as starting point to determine the optimal quantity. In the next sections ‘mu forecast’ will be used as the value for $\mu$ in equation (4.11).

$$Q^* = \frac{EOQ}{\sqrt{1 - \frac{\mu}{h}}}$$  \hspace{1cm} (4.11)

First, the case of a drift $\mu$ equal to 0.1 is observed with a volatility $\sigma$ of zero. For various forecasts of $\mu$, different order quantities are obtained and these give the expected results, depicted in figures 4.5. The inflationary model is also very robust according to the simulations. For the EOQ model, the total holding costs and order cost are equal. Fig. 4.6 shows that the order quantity increases with increasing estimated $\mu$. This leads to bigger differences between holding and order costs. When the drift is underestimated, the order costs are higher because there are more orders which leads to a higher fixed cost. Plus on average, a higher purchase price will be paid per product (fig. 4.7). When the drift is overestimated however, the extra inventory costs will outweigh the small drop in order cost. These cost increases are however very small for forecast errors. The same U-shaped curve of the total cost is obtained for forecast errors of demand.
Table 4.1: Overview of the parameters in simulation of model 1.

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Drift rate $\mu$</td>
<td>0.1</td>
<td>Demand $D$</td>
<td>5000</td>
</tr>
<tr>
<td>Volatility $\sigma$</td>
<td>0.0</td>
<td>Simulation time</td>
<td>5 years</td>
</tr>
<tr>
<td>Unit purchase cost $P_0$</td>
<td>$10</td>
<td>Carrying charge rate $h$</td>
<td>0.4</td>
</tr>
<tr>
<td>Fixed order cost $K_0$</td>
<td>$50</td>
<td>Optimal order quantity $Q^*$</td>
<td>408 products</td>
</tr>
</tbody>
</table>

Figure 4.5: Total cost function in function of forecasted drift rate $\mu$. 
Inflation model with variance

First of all, it can be noted that the distribution of the order quantities is skewed to the left as can be seen in fig. 4.8. More orders are made when volatility $\sigma$ increases. Although the order quantity is generally lower, the average order quantity increases for increasing variance (see framework below). This may explain why higher price variance is generally associated with higher inventory levels. Fig. 4.10 depicts that the optimal order quantity is still equal to the EOQ adjusted for inflation when both
the drift and volatility are non-zero.

Figure 4.8: Distribution of order quantities in function of volatility.
Influence of volatility on average order quantity

In the simple case where $\sigma$ is equal to zero, the order quantity remains always the same, because the fixed cost $K_t$ increases with the same percentage as the purchasing unit price $P_t$. The order quantities are not longer constant for non-zero volatilities. An upward trend in average order quantity can be observed, which can be explained by the fact that an increase in the $K_t/P_t$ ratio leads to a higher order quantity difference than an equivalent decrease. Suppose that $Q_0$ is equal to 500. If the fixed cost increases with 20% for instance, then $Q$ increases to $\sqrt{1+0.2}\times 500$, or 548 ($= +48$). On the other hand, if the purchasing unit price $P_t$ increases with 20%, then $Q$ decreases to $\frac{1}{\sqrt{1+0.2}}\times 500$, or 456 ($= -44$).

Figure 4.9: Influence of volatility on average order quantity for a simulation time of two years (blue) and five years (red).

The optimal order quantity is equally robust for varying volatilities $\sigma$, but the total cost varies much more. Figure 4.11 graphically depicts the distribution of the total cost. For many firms this is an undesirable result because the damage of a very high cost may outweigh the benefit of a very low cost. They prefer a position that neutralizes the risk of a very high purchase cost. Some companies will therefore conclude long-term contracts or hedge their exposure to price changes in order to limit or offset the risk. With the hedge they aim to reduce real costs like taxes, costs of financial distress$^5$ and costs of external finance.

$^5$Financial distress: a term in corporate finance used to indicate a condition when promises to creditors of a company are broken or honored with difficulty. If financial distress cannot be relieved, it can lead to bankruptcy.
4.4. MODEL WITH GBM PRICE EVOLUTION

Figure 4.10: Total cost in function of demand forecast and volatility ($\sigma_{\text{green}} = 0; \sigma_{\text{red}} = 0.25$).

Figure 4.11: Box plots of total cost ($\sigma = 0.15$ (left); $\sigma = 0.10$ (right)).
4.4. MODEL WITH GBM PRICE EVOLUTION

4.4.4 Influence of parameters and cost savings

Influence of inventory cost

As mentioned earlier, more savings can be made when demand and the inventory carrying cost are low, while the fixed cost and the inflation rate are high. Table 4.2 illustrates this for different annual carrying charge rates \( h \). The same parameters are taken as table 4.1 except that the volatility \( \sigma \) is set equal to 0.25.

Table 4.2: Influence of inventory carrying charge rate on total expected cost.

<table>
<thead>
<tr>
<th>( h )</th>
<th>( \hat{E}(TC_{EOQ}) )</th>
<th>( \hat{E}(TC_{inflation}) )</th>
<th>% saving</th>
</tr>
</thead>
<tbody>
<tr>
<td>20%</td>
<td>331158.1</td>
<td>330892.0</td>
<td>0.080</td>
</tr>
<tr>
<td>30%</td>
<td>332855.0</td>
<td>332718.4</td>
<td>0.041</td>
</tr>
<tr>
<td>40%</td>
<td>334213.3</td>
<td>334128.8</td>
<td>0.025</td>
</tr>
<tr>
<td>50%</td>
<td>335378.1</td>
<td>335316.5</td>
<td>0.018</td>
</tr>
<tr>
<td>60%</td>
<td>336401.8</td>
<td>336364.8</td>
<td>0.011</td>
</tr>
</tbody>
</table>

Influence of volatility

The volatility \( \sigma \) has an influence on the average order quantity and total cost as discussed earlier. Fig. 4.12 shows that the average total cost decreases as volatility increases. Prices fluctuate more and the inventory manager is able to take advantage by adapting the quantities in function of the price evolution. A low quantity is ordered when the the unit purchase price \( P_t \) is relatively high compared to the fixed order cost \( K_t \). In the face of rapid price fluctuation, the firms need to be responsive and accurately assess the price opportunities. When there is only volatility and no drift, the classical EOQ model gives the best results. The loss by abrupt price rises are offset by the gains of equally likely price drops.
4.4. MODEL WITH GBM PRICE EVOLUTION

Figure 4.12: Influence of volatility on average total cost. The drift $\mu$ is equal to 0.1.

**Influence of time-value of money**

Up to now, time-value of money has been ignored in the model. As mentioned earlier though, it can affect the optimal policy. If we assume that $\alpha$ is the interest rate, then the present value of the cash flows of an ordering cycle can be given by equation

\[
(PV \text{ ordering costs}) + (PV \text{ inventory costs}) = (K_t + P_tQ)e^{-\alpha t} + hP_t\left[\int_0^T (Q - tD)e^{-\alpha t} dt\right]e^{-\alpha t} \tag{4.12}
\]

which can be simplified to:

\[
(K_t + P_tQ)e^{-\alpha t} + \frac{hP_tD}{\alpha} \left[T + \frac{e^{-\alpha T} - 1}{\alpha}\right]e^{-\alpha t} \tag{4.13}
\]

Note that this equation obtains a larger inventory cost than taking the average inventory cost over the time period and discounting it for $t = T/2$. The inventory cost decreases with time and high inventory costs therefore correlate with low discount factors.

As depicted in fig. 4.13, the optimal order quantities are lower when time-value of money is taken into account. With the interest rate $\alpha$ set equal to 0.2 and the drift equal to 0.1, the optimal $\mu$ in (4.11) is $\mu - \alpha$. When the drift $\mu$ and interest rate $\alpha$ have approximately the same value, the economic order
quantity turns out to be optimal for this first model. Furthermore, it is also shown that ignoring time-value of money or inflation is most critical for low demand products with high order costs and low inventory carrying rates. For these products, the highest savings can be made.

Figure 4.13: Influence of adjusting $\mu$ for the time-value of money. The drift $\mu$ is equal to 0.1 and interest rate $\alpha$ equal to 0.2.

4.4.5 Conclusion

Under the assumption of a price evolution that follows a geometric Brownian motion, the adjusted EOQ model for inflation turns out to be optimal in case of a more stochastic price evolution. Thus volatility has no effect on the optimal policy in this model and the same policy is found as in section 3.2. When the drift $\mu$ and interest rate $\alpha$ have approximately the same value, the economic order quantity is still optimal when calculated for the prevailing price levels. The reason for this, besides that GBM is also exponential in the mean, is that the GBM has no incentive to move to its expected value in the short term. The Markov property and the constant expected return ensure relative lognormal increments independent of the price. In other words, the probability of a price increase of 5% when the price is low is equal to the case when the price is high. A more general price evolution model will be discussed in section 4.5 where price jumps and seasonality is included in the GBM.
4.5 Model with modified GBM

4.5.1 Model

The nature of the GBM model does not reflect the true price movements except in the very short term. In the real world, monetary and fiscal policy changes, exogenous shocks, or other phenomena sometimes lead to wild movements in the product price. Seasonality often plays a part in price evolutions as well. Normal increases and decreases in supply and demand for particular commodities seem to occur every year in fairly consistent patterns. When examining the commodity energy for instance, the following pattern can be observed in the northern hemisphere: a peak demand for natural gas in winter when demand is high for home heating and a smaller spike in the summer when natural gas is used in power generation to meet air conditioning demand. Gasoline, meanwhile, has its peak demand period in the summer when consumers drive to cottages and other vacation destinations.

Ladde and Wu explored the utilization of the GBM model under different data partitioning processes with price jumps and developed a modified geometric Brownian motion model \[17\]. They concluded that data partitioning may give better fits for the stock price datasets than the standard GBM model. Due to the accumulated errors in models without price jumps, models with price jumps turned out to be much better. The quarterly GBM with jumps is obtained as follows:

\[
X_t = \begin{cases} 
X_{t_1}^{q_1} = X_0 e^{(\mu^{q_1} - \frac{\sigma^{q_1}^2}{2})t + \sigma^{q_1}W_t}, & X_0 = X_0 \text{ and } 0 \leq t < t_1 \\
X_{t_2}^{q_2} = \phi_1 X_1 e^{(\mu^{q_2} - \frac{\sigma^{q_2}^2}{2})t + \sigma^{q_2}W_t}, & X_1 = \lim_{t \to t_1} X_{t_1}^{q_1} \text{ and } t_1 \leq t < t_2 \\
X_{t_3}^{q_3} = \phi_2 X_2 e^{(\mu^{q_3} - \frac{\sigma^{q_3}^2}{2})t + \sigma^{q_3}W_t}, & X_2 = \lim_{t \to t_2} X_{t_2}^{q_2} \text{ and } t_2 \leq t < t_3 \\
X_{t_4}^{q_4} = \phi_3 X_3 e^{(\mu^{q_4} - \frac{\sigma^{q_4}^2}{2})t + \sigma^{q_4}W_t}, & X_3 = \lim_{t \to t_3} X_{t_3}^{q_3} \text{ and } t_3 \leq t < t_4 \\
X_{t_5}^{q_5} = \phi_4 X_4 e^{(\mu^{q_5} - \frac{\sigma^{q_5}^2}{2})t + \sigma^{q_5}W_t}, & X_4 = \lim_{t \to t_4} X_{t_4}^{q_4} \text{ and } t_4 \leq t < t_5
\end{cases}
\]

where \( \mu^{q_1}, \mu^{q_2}, \mu^{q_3}, \mu^{q_4} \) and \( \sigma^{q_1}, \sigma^{q_2}, \sigma^{q_3}, \sigma^{q_4} \) are the drift and the volatility for the four quarters respectively. \( \phi_1, \phi_2, \phi_3 \) are the jump coefficients and the jumps are assumed to be at the end of the quarter.
4.5. MODEL WITH MODIFIED GBM

In contrast to the model of Ladde and Wu, the jumps do not always occur at the end of a quarter but can happen at any time. We will denote the probability of a jump by the parameter $\zeta$ and the maximum percentage price increase as $\beta$. Two sample paths of the modified GBM are given in figure 4.14.

![Figure 4.14: Example sample paths of modified geometric Brownian motion with $\mu_{q_1} = 0.35$, $\mu_{q_2} = -0.15$, $\mu_{q_3} = 0.25$ and $\mu_{q_4} = -0.05$. The volatility $\sigma = 0.15$ (blue curve) and $\sigma = 0.05$ (red curve). The green curve shows the expected value in function of $t$.](image)

The green curve gives the expected value of the price at each time instant $t$. A cyclical behavior can be observed and jumps occur for instance after 3.9 and 9.5 years. The price is subject to inflation in quarter one and three, while the rest of the year it is subject to deflation. The annual inflation rate can be calculated as $e^\mu - 1$ with $\mu$ equal to $\sum_{i=1}^4 \mu_i$.

4.5.2 Optimal order policy

Different policies can now be examined:

1. EOQ model which does not take the price evolution into account.
2. EOQ model modified for inflation with the annual inflation rate equal to $\sum_{i=1}^4 \mu_i$.
3. quarterly EOQ, order quantity depends on estimated quarterly inflation rate.
4. adjusted quarterly EOQ, order quantity depends on estimated quarterly inflation rate for the following replenishment cycle length.
Since the drift $\mu$ is not longer the same over the whole year, the inventory manager may have to vary his order decision. Policy 4 dictates to adjust the estimated $\mu$ for replenishments that pass through the next quarter. When the order is made at time instant $t_0 < t_i$ and the replenishment cycle will end at $t > t_i$ with $t$ equal to $t_0 + Q_{t0}/D$ and $t_i$, the end of $q_i$, the estimated $\mu$ is determined by equation 4.14:

$$\mu = \frac{t_i - t_0}{t - t_0} \mu_i + \frac{t - t_i}{t - t_0} \mu_{i+1}$$  \hspace{1cm} (4.14)$$

Because $\mu$ and the order quantity $Q$ depend on each other, an iterative process is done to find the value of $\mu$. An overview of the used parameters is shown in table 4.3. The different policies are compared in fig. 4.15. The lowest cost is obtained by policy 4. Thus a small improvement is found when the average drift rate over the replenishment cycle length is used.

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Drift rate $\mu_{q1}$</td>
<td>0.35</td>
<td>Volatility $\sigma$</td>
<td>0.15</td>
</tr>
<tr>
<td>Drift rate $\mu_{q2}$</td>
<td>-0.15</td>
<td>Fixed order cost $K_0$</td>
<td>$100$</td>
</tr>
<tr>
<td>Drift rate $\mu_{q3}$</td>
<td>0.25</td>
<td>Probability of jump $\zeta$</td>
<td>0.5 times a year</td>
</tr>
<tr>
<td>Drift rate $\mu_{q4}$</td>
<td>-0.05</td>
<td>Carrying charge rate $h$</td>
<td>0.4</td>
</tr>
<tr>
<td>Simulation time</td>
<td>5 years</td>
<td>Unit purchase cost $P_0$</td>
<td>$10$</td>
</tr>
<tr>
<td>Interest rate $\alpha$</td>
<td>0.2</td>
<td>Maximum percentage price jump $\beta$</td>
<td>0.1</td>
</tr>
</tbody>
</table>
So far, ordering was only possible when the inventory was empty, but in a fluctuating market this may be suboptimal as shown in [9]. Under some conditions it may be better to place orders at other time instants too. A policy is proposed where an order may be placed at the end of a quarter or period when the inventory level is lower than a certain threshold. The inventory does not longer have to be empty before an order is placed when the following period is predicted to be subject to a much higher inflation. In the example path of fig. 4.14 this is for instance the case at the beginning of $q_1$ and $q_3$. The price is probably at its lowest point here, and the cost savings of the lower purchase unit price may outweigh the extra cost due to a higher discounted order cost (time-value of money) and extra inventory. When the inventory level is still high at the transition point, a new order leads to excessive inventory levels and corresponding costs. Fig. 4.16 shows that the policy does not lead to lower costs when the interest rate $\alpha$ is higher than a certain threshold value above the average inflation rate. Costs should be delayed as much as possible because of time-value of money. Note that above a certain threshold the policy does not change because it is larger than the order quantities and an order is always placed at the end of relatively low inflation rates, regardless of the inventory level.
4.5. MODEL WITH MODIFIED GBM

Figure 4.16: Comparison of the average total cost in function of the threshold inventory level with an interest rate bigger than the yearly inflation rate.

However, when the interest rate is lower than the inflation rate or below a certain threshold value, cost savings can be made. Fig. 4.17 depicts the total costs for different threshold inventory levels under the assumption that the interest rate \( \alpha \) is equal to 5%. Under the given assumptions, a cost saving of 4% was made.

Figure 4.17: Comparison of the average total cost in function of the threshold inventory level with an interest rate smaller than the yearly inflation rate.
4.5.3 Conclusion

The more general price evolution model, with price jumps and seasonality included, does not lead to very different results. Adjusting the order quantity for inflation leads to the best results. But it should be noted that the order quantity must depend on the expected price evolution of the whole replenishment time. Equation 4.14 calculates with an iterative process the optimal forecasted drift $\mu$ of the period. Furthermore, under (relatively) high interest rates it is better to wait until the inventory is empty in order to keep the holding costs low and delay the order cost, even when it is known that the inflation rate will be much higher in the near future. In the opposite case, early orders can lead to significant cost reductions.

4.6 Model with stochastic demand and delivery lead time

4.6.1 Model

So far only the assumption of constant price level has been relaxed. In this section, the model will be extended by relaxing some other assumptions. When the assumption of instant replenishment is not longer true, the firm cannot wait anymore till the stock is empty. An order needs to be placed when the inventory level is still sufficient enough for the demand during the ordering interval. For constant replenishment lead time $L$, Axsäter states the following simple relationship [1]:

$$IL(t + L) = IP(t) - D(t, t + L)$$  \hspace{1cm} (4.15)

With $D(t, t + L)$, the stochastic demand in the interval $(t, t+L)$ and $IP(t)$, the inventory position at time $t$ or stock on-hand + outstanding orders - backorders. At time $t + L$, everything that was on order at time $t$ has been delivered while orders that have been triggered after time $t$ have not reached the inventory due to the lead-time. This situation is depicted in figure 4.18 where the delivery lead time and demand is taken into account in the determination of the reorder point.
4.6. MODEL WITH STOCHASTIC DEMAND AND DELIVERY LEAD TIME

Furthermore, the assumptions of constant demand and replenishment lead times are dropped. They are replaced by stochastic variables. Due to the uncertainties in delivery lead time a safety stock will have to be maintained in order to attain a certain service level. In case a delivery arrives late, the safety stock will function as a buffer to prevent a production interruption or unmet demand. Fig. 4.19 depicts this more realistic situation with fluctuating demand and stochastic lead time. The time between two orders is not longer constant and known in advance. Note that the inventory position is equal to the sum of on-hand inventory and inventory on order (minus any backorders).

Figure 4.18: Evolution of inventory level with constant delivery lead time and deterministic demand.

Figure 4.19: Evolution of inventory level (green) and inventory position (red) with stochastic demand and lead time.
Demand and delivery lead time distributions

First of all, suitable frequency distributions have to be chosen in order to make an adequate model. For items with a high demand, it is usually more common and efficient to model the demand as a continuous distribution over a period of time. Thus the first task consists of finding a suitable continuous distribution. Burgin states that the normal and exponential distributions are incapable of adequately describing the demand characteristics of all items found in the typical inventory \cite{5}. The normal distribution is defined for negative values which is unrealistic for delivery lead time and demand. And because of its symmetry it is also only adequate for representing the demand of very fast-moving items. On the other hand the exponential distribution is monotonic decreasing and is only a good way to represent the demand of slow-moving items.

In our simulation, the gamma distribution will therefore be used to model the daily demand and lead time of orders. One of the main advantages of a gamma distributed demand and delivery lead time is that it is only defined for non-negative values. Furthermore, depending on the values of the parameters, the distribution can be monotonic decreasing, normally distributed (as the modulus tend to infinity) or unimodal distributed and heavily skewed to the right. The positive skewness, which only depends on the shape parameter $\kappa$, is an interesting characteristic because it seems a reasonable assumption for the lead time and demand. When unforeseen events occur, the lead time is more likely to be very high than very low. If the bulk of the orders are delivered between four and eight days for instance, it is more likely to have an order lead time of 11 days than a lead time of one day. Finally, we assume that events, such as order delivery or product demand, may only happen at discrete time points. Therefore their values are rounded to the nearest integer. In fig. 4.20 the parameters of $\Gamma_L(\kappa_L, \theta_L)$ are set equal to $\kappa_L = 3.5$ and $\theta_L = 2$, which results in an average lead time of 7 days. For the normal distribution $N(\mu, \sigma)$ the equivalent parameters $\mu$ and $\sigma$ are respectively equal to 7 and 3.74.
The parameters of $\Gamma_D = (\kappa_D, \theta_D)$ are set equal to $\kappa_D = 7$ and $\theta_D = 2$, which results in an average daily demand of 14 products. This is depicted in fig. 4.21. The parameters of the gamma distribution are chosen in such way that the average demand is about the same as in the previous simulations in order to make a comparison possible with the results of the other sections.
4.6. MODEL WITH STOCHASTIC DEMAND AND DELIVERY LEAD TIME

Stock-out cost

Various reasons can lead to an unexpected long lead time or high demand. When this happens it is possible that the safety stock is depleted and demand can no longer be fulfilled. In this case there are two possibilities: the demand can be either lost or backlogged. The former results in a sales opportunity cost, the firm lost its margin on the product. In the case of backlog, the demand is not lost but the profit margin is possibly decreased because penalties may have to be paid to the customer or because of extra costs caused by a rush order. Besides, there is also the risk that the reputation of the company is damaged. Customers may switch to a competitor next time and thus the future profit may be reduced. The total cost of a stock-out is therefore very difficult to estimate.

In our model, the case of backlogging is studied. Thus the inventory position will be adapted for the backlog. Unmet demand is backordered and fulfilled as soon as a replenishment arrives. A fixed cost is assumed for backlogging which does not depend on the time. Meaning that a sales order delivered only one day too late costs for instance as much as an order which is delivered five months too late. Furthermore, in-transit inventory, the products that have been shipped but not yet received, is assumed to be in the possession of the supplier and no inventory costs are considered unto delivery. The order cost are paid at delivery and discounted at delivery time.

4.6.2 Optimal order policy and service level

The problem of setting the safety stock when both the demand and lead time are random variables is discussed in [12]. They show that the standard procedure for known parameters, presented in the literature, can yield results that are far from the desired result. This is the case when the random variable lead time demand is not normally distributed. In the simulation the same formula as the standard procedure is used to calculate the reorder point $R$ (eq. 4.16) where the $z$-value varies depending on the desired service level. The attained service level is however estimated by the simulation. We define the $\alpha$ service level as the probability of no-stock-out per replenishment and the fill rate as the proportion of total demand met from stock on hand. The parameters of the simulation are given in table 4.4.

$$R = \bar{D} \bar{L} + z \sqrt{\bar{L} \sigma_D^2 + \bar{D}^2 \sigma_L^2}$$  \hspace{1cm} (4.16)
4.6. MODEL WITH STOCHASTIC DEMAND AND DELIVERY LEAD TIME

Table 4.4: Overview of the parameters in simulation model 3.

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Drift rate ( \mu^1 )</td>
<td>0.35</td>
<td>Volatility ( \sigma )</td>
<td>0.15</td>
</tr>
<tr>
<td>Drift rate ( \mu^2 )</td>
<td>-0.15</td>
<td>Demand shape factor ( \kappa_D )</td>
<td>7</td>
</tr>
<tr>
<td>Drift rate ( \mu^3 )</td>
<td>0.25</td>
<td>Demand scale factor ( \theta_D )</td>
<td>2</td>
</tr>
<tr>
<td>Drift rate ( \mu^4 )</td>
<td>-0.05</td>
<td>Delivery lead time shape factor ( \kappa_L )</td>
<td>3.5</td>
</tr>
<tr>
<td>Simulation time</td>
<td>5 years</td>
<td>Delivery lead time scale factor ( \theta_L )</td>
<td>2</td>
</tr>
<tr>
<td>Unit purchase cost ( P_0 )</td>
<td>$ 10</td>
<td>Carrying charge rate ( h )</td>
<td>0.4</td>
</tr>
<tr>
<td>Fixed order cost ( K_0 )</td>
<td>$ 50</td>
<td>Jump parameter ( \zeta )</td>
<td>0.5 times a year</td>
</tr>
<tr>
<td>Penalty cost ( F_B )</td>
<td>$ 2</td>
<td>Interest rate ( \alpha )</td>
<td>0.2</td>
</tr>
</tbody>
</table>

The variance of a gamma distribution can be calculated as: \( \kappa \theta^2 \). Thus the variance increases with increasing \( \theta \). In fig. 4.22, it is shown that a system with higher variance (grey curve) approaches the desired service level of (4.16) better for low \( z \)-values. The higher variances \( \sigma_D \) and \( \sigma_L \) lead to a higher reorder point \( R \) and provide enough buffer to cope with the demand and lead time uncertainty. When high service levels are desired however, formula (4.16) does not provide good results in every case. A higher safety stock has to be taken in order to obtain the desired service level.
4.6. MODEL WITH STOCHASTIC DEMAND AND DELIVERY LEAD TIME

Figure 4.22: Attained service level for different delivery lead time and demand parameters.

When considering the fill rate, different factors may have an influence on the result. Figures 4.23 and 4.24 show that the fill rate increases for increasing order quantity and \( z \)-value. When the fixed order cost \( K \) or drift \( \mu \) is high, the order quantity will be higher and more demand will be met on time. In contrast to the \( \alpha \) service level, large orders and corresponding high cycle stock increase the fill rate of a component. This is the reason why the fill rate measure is widely used in industrial practice. It does not only reflect the stock-out event but also the amount of backorders.
4.6. MODEL WITH STOCHASTIC DEMAND AND DELIVERY LEAD TIME

The optimal order quantity and corresponding service level can only be determined by taking all the factors into account. The higher the penalty cost $F_B$, the higher the optimal service level will be for instance. In fig. 4.25, the cost for a backorder $F_B$ is assumed to be equal to the unit purchase price. Note that while determining the optimal policy, both the $z$-value and order quantity need to be taken into account simultaneously. Determining the optimum of a factor one at a time may result in a worse outcome than varying them simultaneously. When the order quantity is set equal to the EOQ, an optimal $z$-value of 1.5 is found. This leads in the next phase to the decision of a forecast drift $\mu$ of 0.1. Clearly, the one-factor-at-a-time method has failed here because it fails to detect the interaction...
4.6. MODEL WITH STOCHASTIC DEMAND AND DELIVERY LEAD TIME

between the \( z \)-value and order quantity. In the following section, the influence of different parameters on the optimal order policy and total cost will be examined.

![Graph showing total cost in function of service level (z-value) and order quantity.]

**Figure 4.25**: Total cost in function of service level (z-value) and order quantity.

### 4.6.3 Influence of parameters and cost savings

Now, the policies that are discussed in section 4.5 will be investigated for the new model. First of all the reorder points for each policy are determined in such way they all attain a fill rate of 99%. Fig. 4.26 depicts that policy 4 is found to be optimal again. The total costs that are found are of course a bit higher than in section 4.5. This can be explained by the extra inventory costs of the safety stock and the backlog penalties. The possible cost reduction again turns out to be small.
4.6. MODEL WITH STOCHASTIC DEMAND AND DELIVERY LEAD TIME

When extra orders are made at the beginning of the quarter, the fill rate increases because the inventory is higher. This kind of policy turns out to be no improvement in case of a relatively high interest rate. Costs should be delayed as much as possible. The opposite is again true for relatively high inflation rates. Fig. 4.27 depicts the case of a high interest rate.

---

**Figure 4.26:** Comparison of the average total cost in function of the order policy.

**Figure 4.27:** Comparison of the average total cost in function of the threshold inventory level with an interest rate bigger than the yearly inflation rate.
4.6.4 Conclusion

In this section, stochastic demand and lead times were included in the model. Orders were not delivered instantaneously anymore. When the stock-out cost is measured, an optimal inventory system can be determined with a corresponding fill rate. The optimal safety stock level will depend on multiple factors such as stock-out cost and order quantity for instance. Finally, the same remarks can be made about the optimal order policy as in section 4.5.
Chapter 5

Markov Decision Process

5.1 Introduction

In chapter 3 and 4, we discussed the impact of inflation on the economic order quantity. The price of a commodity can also have a downward trend by factors such as political and regulatory changes, new technology or changing market conditions. They may affect the supply and demand of the commodity and whenever supply exceeds demand there will likely be downward pressure on the price. In this section it is assumed that the price evolution of the purchasing goods can move between an upward and a downward trend. The Markov decision process is stationary in the long-term because the probabilities and price differences remain the same the entire time. When observing a small time window however, it looks like a non-stationary price evolution, either going upward or downward. Different order policies can be observed under the different market conditions.

In financial markets the terms bull market and bear market describe upward and downward market trends, respectively. The former is associated with increasing investor confidence, and increased investing in anticipation of future price increases. While the latter is a general decline because of widespread investor fear and pessimism. The same terms will be used here to describe the two kinds of price evolution. A bull market will be associated with increasing prices, while a decreasing market will be referred to as a bear market. This terminology is purely used to facilitate the explanation of the model. It should be noted that downward trends do not generally coincide with recessions. As explained below, prices do not always decline during a recession. A downward trend rather occurs at times where prices decrease because of innovation or improved technical conditions for instance.

Markov decision processes will be used in this section. Markov decision processes are powerful an-
5.2. MODEL DESCRIPTION

Analytical tools that have been widely used in many industrial and manufacturing applications such as logistics, finance and inventory control. They can be an elegant way for solving stochastic, dynamic programs under some limiting assumptions. In section 5.2 the modeling of the problem will be explained and the Markov decision process will be developed. Furthermore, the solution method will be discussed in 5.3. And finally, in section 5.4 the main results are presented.

Why prices do not decline during a recession

Recessions are part of the business cycle (see fig. 5.1). During recession, demand for goods declines and one would expect the prices to go down due to the law of demand and supply. This does not happen in actuality however. Inflation rates fall during a recession, but they generally do not go below zero. The reason for this is that the money supply is constantly expanding, for instance through fractional reserve banking. Hereby, the economy is subject to consistent inflationary pressure. And despite the real GDP falls, the CPI rises.

Figure 5.1: Stages of the classic business cycle

5.2 Model description

As in section 4.4 and 4.5, two types of costs will be considered in the analysis of the inventory system. These are the replenishment cost and the holding cost. In our simplified model, shortages can not occur because we assume instantaneous delivery. Hence, the shortage cost is not taken into consideration. The purchase cost also has to be taken into account, because the product price is not assumed to be constant. In this section, the assumptions of the model are explained and the influence of some parameters are studied.
5.2.1 Demand fluctuations

Demand can fluctuate throughout time. Under the situation of uncertain demand, different policies may be applied to improve the performance of the lot sizing rules. This includes the determination of the safety stock. Safety stock policy is normally applied where there is uncertainty in demand and/or supply during the lead time. In this study the lead time is assumed to be zero, hence a safety stock is not required. Because the problem is modeled as a discrete-time Markov chain (DTMC), a discrete probability distribution is required. At discrete time units, the demand will be either equal to one or zero, dependent on the value of two parameters, \( p \) and \( r \). Demand is assumed to happen in \( r \) so-called stages (or ‘phases’) that have to be traversed one after the other. After reaching the last stage \( r \), an item is taken from inventory and the stage is again in its initial state. Concerning the demand fluctuations, it is therefore assumed that:

- the time between two consecutive stages is geometrically distributed with parameter \( p \),
- after \( r \) ‘successes’, the demand is on its final level and the inventory level decreases with one unit,
- the inter-arrival time between successive demands is negative binomially distributed with the number of stages \( r \) and probability parameter \( p \). This means that at each discrete time unit, demand either moves on to the next stage or remains at the same stage with probability \( p \) and \( 1 - p \) respectively.

This is illustrated by fig. 5.2. The inventory level lowers randomly by the negative binomial distribution.

Figure 5.2: Example sample path of demand with negative binomial distribution NB(4; 0.7). After four successes the inventory level decreases with one unit and demand is again in its initial stage.
The demand model is described as a Markov chain and fig. 5.3 depicts this graphically. The maximum number of products in inventory is equal to the capacity $C$. The assumption is made that an order can only be placed when the inventory is empty. The zero-state is an absorbing state and without any action the system would remain in this state. The decision to order will be made at discrete time slots, e.g., daily or weekly, right after the moment the last item in inventory is used. After the order is placed, the new state depends on the order quantity and the subsequent ‘success’ of the demand phase. The dotted line gives an example of an order of $C - 1$ items with immediate ‘success’.

Figure 5.3: Graphic representation of inter-arrival times between two consecutive demands.

Figure 5.4 represents the inter-arrival times between two consecutive demands in function of $r$. The mean inter-arrival time is for all distributions equal to 10, but the probability distribution and variance clearly vary with $r$. The variance of a negative binomial distribution can be calculated by equation 5.1. The variance of the red, blue and black distribution is for example equal to respectively 90, 40 and 10. In case there are $r = 10$ possible stages, the inter-arrival time would be constant and equal to 10 ($\sigma^2 = 0$). Furthermore, note that the probability distributions are skewed to the right, which means...
they have longer tails on the right side. [10] states that there is empirical evidence to justify the use of the negative binomial distribution for many items.

\[ \text{variance} = \frac{p^r}{(1-p)^2} \]  

\( (5.1) \)

**Figure 5.4:** Probability mass functions of the time \( t \) between two consecutive demands with mean 10 and different numbers of intermediate stages \( r \).

### 5.2.2 Price fluctuations

Besides demand fluctuations, price fluctuations are considered to influence the inventory system of a product as well. The price fluctuations will also be described as a Markovian chain. The market can be in two different conditions: the price can be either subject to an upward or a downward trend, respectively referred to a bull market and bear market.

In a bull market, the chance of a price increase (\( \alpha \)) is higher than a price decrease (\( \beta \)). While in a bear market, the opposite is true (\( \lambda \) will be higher than \( \gamma \)). The transition probabilities between the two different trends are chosen in such way that it is approximately equal to the inverse of the average time of a bull/ bear cycle, the average time needed to reach the highest state. Furthermore, the market bottoms or tops should not last too long. This behavior and the values for the parameters will be further discussed below. An overview of the parameters is given in table 5.1.
Table 5.1: Overview of parameters in the model.

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>$p$</td>
<td>Probability of ‘success’ (Negative binomial)</td>
</tr>
<tr>
<td>$r$</td>
<td>Number of demand stages (Negative binomial)</td>
</tr>
<tr>
<td>Cap</td>
<td>Capacity of inventory storage</td>
</tr>
<tr>
<td>$\alpha$</td>
<td>Probability of price increase in bull market</td>
</tr>
<tr>
<td>$\beta$</td>
<td>Probability of price decrease in bull market</td>
</tr>
<tr>
<td>$\gamma$</td>
<td>Probability of price increase in bear market</td>
</tr>
<tr>
<td>$\lambda$</td>
<td>Probability of price decrease in bear market</td>
</tr>
<tr>
<td>$\epsilon$</td>
<td>Probability of transition bull market to bear market</td>
</tr>
<tr>
<td>$\delta$</td>
<td>Probability of transition bull market to bear market</td>
</tr>
<tr>
<td>$v$</td>
<td>Factor for tuning transition probability</td>
</tr>
</tbody>
</table>

Fig. 5.5 depicts the possible transitions when the system is in a *bullish* state $k$. There are six possible states to which the market can move to. For the next time step, the price may fall, rise or stay the same and possibly accompanied by a transition to a bear market. The probability that the price level $k$ remains unchanged is equal to $1-(\alpha+\beta+3\delta_k)$. The number of price levels is the same for both bull and bear markets and denoted by $n$. The relation between the price levels is given by the following recursive formula:

$$P_{k+1} = P_k (1 + i), \quad k = 1, ..., n; \quad P_1 = 10 \quad (5.2)$$
5.2. MODEL DESCRIPTION

**Remarks:**
- $\alpha$ greater than $\beta$
- $\lambda$ greater than $\gamma$
- $\delta$ and $\epsilon$ very small
- $\omega_k = 1 - \alpha - \beta - 3 \delta_k$

**Figure 5.5:** Graphic representation of price fluctuations at level $k$. The transition probability in the market bottom is high for the transition of a bear market to bull market and low for the opposite direction. The converse is true for a market top. The probability that the price level moves further away from its long-term average decreases and the transition probability is highest for a market top (or bottom in case of bear market). As a result the market bottoms or tops do not last too long. To model this increasing chance of transition, it is opted to increase the probability with a factor $v (< 1)$. If the price level approaches to its top, the probability of transition to a bear market becomes bigger in a bullish market.

$$\delta_{k+1} = \frac{\delta_k}{v} \quad k = 1, ..., n; \quad \delta_1 = \delta \cdot v^n \quad (5.3)$$

$$\epsilon_{k+1} = \epsilon_k \cdot v \quad k = 1, ..., n; \quad \epsilon_1 = \epsilon \cdot v \quad (5.4)$$

---

6 Market bottom: trend reversal and end of a market downturn, precedes the beginning of an upward moving trend.
7 Market top: market has reached highest point for some time, the end of a market upturn.
5.2.3 Parameter values

In the previous section, a few characteristics of the price evolution model are listed:

- price increases (drops) are more likely than price drops (increases) in bull (bear) markets.
- the evolution must have a triangular wave pattern, i.e. market bottoms or tops must not last too long.
- market transitions are more likely near market tops than market bottoms in a bull market.
- market transitions are less likely near market tops than market bottoms in a bear market.

It is important for the model that the value of each parameter is chosen wisely. Analytical considerations as well as simulations are used to determine these parameter values. We first determine the average of the time it would take to go from the lowest to the highest price level in the bull market \( E[K] \), assuming transitions to the bear market states are impossible. This time duration \( K \) has a phase-type distribution, with the lowest bull price level as the starting state and with the highest bull price level as the absorbing state. The discrete phase-type distribution \( \text{PH}(\tau, T) \) is described in [18] and the moments of the distribution are given by the following equation:

\[
E[K(K-1)...(K-k+1)] = k! \tau (I - T)^{-k} T^{k-1} \mathbf{1}
\]  

(5.5)

with \( I \) equal to the identity matrix and \( \mathbf{1} \) the all-ones vector.

\( \tau \), a \( 1 \times n \) vector with the starting probabilities, is equal to \([1 \ 0 \ 0 \ ... \ 0]\)

\[
\begin{pmatrix}
\alpha & -\alpha & 0 & 0 & 0 & ... & 0 \\
-\beta & \alpha + \beta & -\alpha & 0 & 0 & ... & 0 \\
0 & -\beta & \alpha + \beta & -\alpha & 0 & ... & 0 \\
0 & 0 & 0 & ... & 0 & -\beta & \alpha + \beta
\end{pmatrix}
\]

and \( T \), the \( n \times n \) transition rate matrix:

To determine the mean, \( k \) in (5.5) is set equal to one and this leads to equation (5.6):

\[
E[K] = \tau (I - T)^{-1} \mathbf{1}
\]  

(5.6)
If \( v \) is set equal to one, the transition probability can be approximately derived by equation [5.7]. \( \zeta \) is equal to the probability that after \( E[K] \) time units, no transition has occurred. Its value ranges between zero and one and clearly depends on \( \delta \), \( \alpha \) and \( \beta \).

\[
(1 - 3\delta)^{E(K)} = \zeta \iff \delta = \frac{1 - \zeta}{3^{E(K)}}
\]

(5.7)

The average time of a bull/ bear cycle \( T_{avg} \) for some parameter values and the resulting ideal \( \delta \) values are given in table 5.2. Two values are considered for \( \zeta \), 10 and 50%. Note that if \( \zeta \) is equal to 0.5, the transition occurred in half of the cases after the market top is reached.

**Table 5.2:** Average time of bull/ bear cycle and calculation of deltas in bull market.

<table>
<thead>
<tr>
<th>( \alpha )</th>
<th>( \beta )</th>
<th>( n )</th>
<th>( E(K) )</th>
<th>( \delta_{\zeta=0.5} \times 10^{-4} )</th>
<th>( \delta_{\zeta=0.1} \times 10^{-4} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.02</td>
<td>0.01</td>
<td>10</td>
<td>900.1</td>
<td>2.57</td>
<td>8.51</td>
</tr>
<tr>
<td>0.02</td>
<td>0.005</td>
<td>10</td>
<td>644.4</td>
<td>3.58</td>
<td>11.9</td>
</tr>
<tr>
<td>0.02</td>
<td>0</td>
<td>10</td>
<td>500</td>
<td>4.62</td>
<td>15.3</td>
</tr>
<tr>
<td>0.05</td>
<td>0.01</td>
<td>10</td>
<td>243.8</td>
<td>9.46</td>
<td>31.3</td>
</tr>
<tr>
<td>0.02</td>
<td>0.01</td>
<td>4</td>
<td>501.6</td>
<td>4.60</td>
<td>15.3</td>
</tr>
<tr>
<td>0.02</td>
<td>0</td>
<td>4</td>
<td>300</td>
<td>7.69</td>
<td>25.5</td>
</tr>
</tbody>
</table>

On the other hand, simulation is also used to study the behavior. 50 runs of 100 years are carried out to simulate the price evolution. Figures [5.6], [5.7] and [5.8] give the estimated probability mass distributions of the price levels for some different parameter sets. The red curve is an example of a bad model. It shows that the obtained values of formula [5.7] do not simply give the desired outcome. The low transition probabilities result in long stays in the market top and bottom in case of respectively a bull market and bear market. This can be explained by the fact that many transitions occur before reaching the highest price level and thus reduce the probabilities of intermediate price levels.

For the blue curve, the transition parameters are increased. Each price level will occur an equal number of times over a long period of time, but there is a high transition rate. The transitions occur on average after only 71 time slots instead of the 900, calculated by equation [5.6] or after 8 % of the steps.
The first two examples show that there is a need for more fine-tuned parameters, and this is where \( v \) emerges. By lowering its value, the probability of early transitions reduces significantly and the desired price evolutions are obtained. The green and purple curves meet the stated goals and will be used to model the system. Finally, fig. 5.9 depicts an example of such a sample path with the parameters of the purple curve. Early transitions are far less likely and the price fluctuate more balanced. The parameters are given in table 5.3.

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Blue curve</th>
<th>Red curve</th>
<th>Purple curve</th>
<th>Green curve</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \alpha = \lambda )</td>
<td>0.02</td>
<td>0.02</td>
<td>0.02</td>
<td>0.05</td>
</tr>
<tr>
<td>( \beta = \gamma )</td>
<td>0.01</td>
<td>0.01</td>
<td>0.005</td>
<td>0.005</td>
</tr>
<tr>
<td>( \epsilon = \delta )</td>
<td>0.005</td>
<td>0.001</td>
<td>0.005</td>
<td>0.025</td>
</tr>
<tr>
<td>( v )</td>
<td>1</td>
<td>1</td>
<td>0.6</td>
<td>0.5</td>
</tr>
<tr>
<td>#steps / ( T_{\text{avg}} )</td>
<td>0.08</td>
<td>0.41</td>
<td>0.74</td>
<td>0.73</td>
</tr>
</tbody>
</table>

Figure 5.6: Probability mass distributions for some different parameter sets in the bull market.
5.2. MODEL DESCRIPTION

Figure 5.7: Probability mass distributions for some different parameter sets in the bear market.

Figure 5.8: Total probability mass distributions for some different parameter sets.
5.2. MODEL DESCRIPTION

Figure 5.9: Example sample path of price evolution with standard values for the parameters: \( v = 0.6, \alpha = 0.02, \beta = 0.005 \) and \( \delta = 0.005 \). \( P_0 \) is equal to 10 dollars and the inflation rate \( i \) is set equal to 5%.

5.2.4 Model

Finally, the model is obtained by combining the two submodels. This results in a three-dimensional discrete-time Markov chain. \( S_{j,k,l} \) represents for instance the state with an inventory of \( C - j \) products, price level \( k \) and demand stage \( l \).

To determine an optimal ordering policy we need to take both immediate and future costs into account. The cost matrix includes the inventory cost, purchase cost and fixed order cost based on the following total cost function:

\[
\text{Cost} = \begin{cases} 
  c_h \, IL, & \text{if inventory level } IL > 0, \\
  K + Q \, P_t, & \text{if inventory level } IL = 0. 
\end{cases}
\]

The inventory cost is proportional to the level of inventory. Note that \( c_h \) is here assumed to be the inventory cost of one time unit. When a time unit is considered to be one day, \( C_h \) is for example equal to \( c_h \times 365 \). For each order a fixed order cost \( K \) and variable purchase cost \( Q \, P_t \) is charged. In contrast
with the inventory costs, the purchase cost depends on the price and the order quantity but not on the current inventory level.

The decision maker now needs to find an optimal policy which dictates the optimal order quantity in each state. It is chosen to base this decision on the minimum expected total discounted cost criterion, since this is suitable for an infinite horizon. In this simplified model, we assume that orders are only possible for empty stocks. The future costs are discounted with a discount factor $d$.

### 5.3 Solution methods

Markov decision processes can be solved by linear programming or dynamic programming. In this study the latter approach is chosen. To summarize briefly, the MDP model consists of the following elements:

- a finite set of states $S$, with $S_{j,k,l}$ representing the state with an inventory of $C - j$ products, price level $k$ and demand stage $l$.
- a finite set of actions $A$, with $Q_s$ representing the action of ordering $Q$ items in state $S_{C,k,l}$.
  Thus an order is only possible when the inventory level $IL$ is equal to 0.
- a probability matrix $P_{Q}(s,s') = Pr(s_{t+1} = s'|s_t = s, Q_s = Q)$, or the probability that ordering $Q$ items in state $S_{j,k,l}$ at time $t$ will lead to state $S_{j',k',l'}$ at time $t + 1$.
- a cost matrix $R_{Q}(s,s')$ equal to the cost of the state at each time instant, with the cost function as in section 5.2.4.

The transformation from $(j, k, l)$ to $(j', k', l')$ is described in figures 5.3 and 5.5 and $j'$ is either equal to $j - Q$ or $j - Q + 1$, depending on whether there is a demand order for the subsequent time instant. Note that the transition matrix also depends on the order quantity.

We use the available ‘MDP Toolbox’ in Matlab to determine the optimal policies. [9] give an overview of the possible solution methods and have shown that the required CPU time does not constitute an obstacle for solving small problems, but increases exponentially with increasing number of states. Therefore a low demand rate, on average one demand in 10 time units, is assumed in order to have relatively low order quantities and number of states. It is also concluded that the ‘modified policy iteration algorithm’ turns out to be the most efficient solution method.
5.4 Numerical results

This section presents the results of the MDP model. The influence of some parameters on the optimal order quantity is studied and discussed.

Influence of market conditions

The expected result of this model is a policy that orders a higher quantity in the case of an upward trend than in the case of a downward trend. When a price increase is more (less) likely than a price decrease, the order quantity will be higher (lower) than the EOQ. The decision maker will order more (less) to anticipate a possible price increase (drop). To start, \( v \) is set equal to 1 and the parameters are chosen as in table 5.4. The probability values are the same as the red curve, i.e. \( \alpha \) and \( \lambda \) equal to 0.02, \( \beta \) and \( \gamma \) equal to 0.01 and \( \epsilon = \delta = 0.001 \). Although the markets remain long in their extrema, this model can give us the first insights of the model. An order policy such as depicted in figure 5.10 is obtained. Note that the EOQ is equal to 39 and a discount factor of 1.0 is assumed. There are four general observations that can be observed:

- The order quantity is higher than the EOQ at market bottoms for both market conditions. Thus the optimal order quantity is high when the price is low.

- The order quantity is lower than the EOQ at market tops for both market conditions.

- In between, the order quantity is higher and lower than the EOQ for respectively, a bull market and bear market.

- The order quantity decreases with increasing price.
5.4. NUMERICAL RESULTS

Table 5.4: Overview of the values for the different parameters in the MDP model.

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\alpha = \lambda$</td>
<td>0.02</td>
<td>$i$</td>
<td>5%</td>
</tr>
<tr>
<td>$\beta = \gamma$</td>
<td>0.01</td>
<td>$K$</td>
<td>$90$</td>
</tr>
<tr>
<td>$\epsilon = \delta$</td>
<td>0.001</td>
<td>$p$</td>
<td>0.3</td>
</tr>
<tr>
<td>$P_0$</td>
<td>$10$</td>
<td>$v$</td>
<td>1</td>
</tr>
<tr>
<td>$d$</td>
<td>1.0</td>
<td>$n$</td>
<td>10</td>
</tr>
<tr>
<td>$c_h$</td>
<td>$0.012$</td>
<td>$r$</td>
<td>3</td>
</tr>
</tbody>
</table>

Figure 5.10: Order policy for a bull market (brown) and a bear market (black).

Fig. 5.11 shows the order policies for different inflation rates. Note that in case there is no inflation, the classic EOQ is ordered for all price levels. The order quantities increase with increasing inflation in the bull market, while they decrease in the bear market.
Figure 5.11: Order policy for a bull market in function of the inflation rate.

When the probabilities of the price evolution model change, other policies are obtained. Market 1 corresponds with the the probabilities of the green curve, i.e. a high $\alpha$ and $\lambda$. Market 2 has probabilities equal to the purple curve and has shown to have a nice triangle-wave pattern for the price level. Finally, order policy 3 is equal to the one obtained in fig. 5.10 because market 3 also corresponds with the red curve.

Figure 5.12: Order policy for a bull market in function of market characteristics.
5.4. NUMERICAL RESULTS

The graphs show a clear difference in order policy. For our standard market conditions (market 2), the same trends are observable but they are more pronounced. There are bigger differences in order quantity for the distinct price levels. The results of market 1 are remarkable. The order quantities for the highest price levels are much lower in comparison with the EOQ and the other policies for both a bull and a bear market. This behavior is caused by the high $\alpha$ and $\lambda$ values. The chance of a drastic market/price change is high after a big order. As the results show, a relative small change in the parameters, may give a totally different policy. Thus the order policy should be customized for the market tendencies of the materials of each company.

**Influence of fixed order cost, demand rate and inventory cost**

The fixed order cost $K$ and the demand rate $p$ both have an equal impact on the order policy. Increasing the parameter, increases the order quantity for both market trends. Here, only the influence of the fixed order cost will be discussed. The standard parameters are set as in table 5.4, except from now on the price evolution will follow the purple curve, i.e. $\alpha = \lambda = 0.02$, $\beta = \gamma = 0.005$, $\delta = \epsilon = 0.005$, $v = 0.6$.

Figures 5.14 and 5.15 depict a few order policies in function of the fixed order cost and the same trends can be observed for $p$. For a fixed order cost of 45, 90 and 180, the EOQ is respectively 27, 39 and 55.

A special case occurs when $K$ is 0. The EOQ is equal to one in that case and the order policy remains unchanged until the inflation rate reaches a threshold. For this specific instance: when the inflation
rate exceeds 6%. Extra products are solely ordered to anticipate the possible large price increase. The fixed cost does not have to be spread over as much items as possible anymore. For bearish markets the order quantity remains of course equal to 1.

Figure 5.14: Order policy for a bull market in function of the fixed order cost $K$.

Figure 5.15: Order policy for a bear market in function of the fixed order cost $K$.

The inventory cost has an opposite influence and lower quantities are ordered for higher inventory costs. When both inventory and fixed order cost are multiplied by the same factor, the EOQ is unchanged, but the order policy changes. The inflation rate has a lower impact when the holding and fixed order cost increase. The order quantities are closer to the economic order quantity and there is less difference between the price levels. Decreasing $P_0$ has of course the same influence and also re-
5.4. NUMERICAL RESULTS

duces the relative importance of inflation. There can be referred to section 3.2 and equation 3.1 where the classical EOQ is also ordered below a certain critical value of $i$.

![Figure 5.16: Order policy for a bear market in function of the fixed order cost and inventory cost (EOQ = 39).](image)

**Influence of discount factor**

Future cash flows are discounted with a discount factor $d$ in order to make future costs weigh less than immediate costs. This discount factor is made of two components, namely the discount rate and time. As expected, lower order quantities are obtained for decreasing discount factor because the purchase cost will be delayed as much as possible.

![Figure 5.17: Order policy for a bull market for different discount factors.](image)
How the discount rate is calculated is thus a crucial factor because a difference of just one or two percentage points in the cost of capital can make a big difference in a company’s order policy. A wide variety of methods can be used to determine discount rates, but there is still a lot of discussion about it in the financial world. The discount factors that are used in fig. 5.17 can be denoted by the mathematical expression $1/(1+\alpha)^t$. With $t$ assumed to be equal to one day (or 1/365 of a year) and $\alpha$ equal to 10% and 5%, respectively, a discount factor of 0.99971 and 0.99986 is found.

**Influence variance of inter-arrival time demand**

In section 5.2.1, the parameter $r$ was introduced to describe a negative binomial distribution. Fig. 5.4 presented the probability mass distributions for different sets of parameters. It was shown that for increasing number of stages, the variance of the inter-arrival time reduces. Now, the influence of demand fluctuations is considered and it can be seen that there is a small increasing trend for decreasing variance in a bull market. The opposite is true in a bear market. Research has shown that this influence is higher for systems with low holding and fixed costs.

![Figure 5.18: Order policy for a bull market in function of demand variability:($n = 8$, $K = 15$).](image)

**Influence of capacity constraint**

Finally, the capacity constraint can also have an impact on the order policy. The order quantity can not exceed the capacity $Cap$ and the decision maker orders $\min(Cap, Q)$. But in some cases the order
quantity raises in a bear market. For some price levels, a higher quantity is ordered in case of the capacity constraint than would be without the constraint. The order quantity is closer to the economic order quantity. This can be explained by the fact that the system will speculate less on a price decrease, because it will not be able to fully benefit from it. When the price is at its lowest point, it will not be able to perform “forward buying”.

Figure 5.19: Order policy for a bear market for different inventory capacities.

5.5 Statistical analysis of results

In the previous section we discussed the impact of some parameters on the optimal order policy. The model of section 5.2 gives order policies which are different from the adjusted EOQ model. In this section we compare the total costs of the different order policies and look if the hypothesis of equal cost can be rejected.

---

*Forward buying: retailers purchase more units than needed during a particular period, hold some of them in inventory, and then sell them in subsequent periods. This phenomenon is often a consequence of trade promotions or anticipation on shortages.*
5.5. STATISTICAL ANALYSIS OF RESULTS

Parameter estimation

The different order decisions are compared on the basis of simulation. To estimate the total cost \( TC \), 100 long trajectories\(^9\) are investigated. The system is stationary ergodic and this allows us to use sample means to determine a consistent and unbiased estimator of the mean.

The system starts from a fixed inventory level and price level. Therefore, we consider the first part of the simulated trajectory as a warm-up period to avoid bias. Once the effect of the initial state has disappeared, the transient period is assumed to be over. An important key in statistical inference is the estimation of the variance. To determine the variance the method of batch means is used. The long trajectory is partitioned into \( n \) periods of \( M \) days and the average statistic for each period is computed. Equations 5.8 and 5.9 give respectively the estimate of the difference in total cost over one period and the standard deviation.

\[
\overline{d} = \frac{1}{n} \sum_{j=1}^{n} d_j \quad (5.8)
\]

\[
s_d^2 = \frac{\sum_{j=1}^{n} (d_j - \overline{d})^2}{n-1} = \frac{\sum_{j=1}^{n} d_j^2 - \frac{(\sum_{j=1}^{n} d_j)^2}{n}}{n-1} \quad (5.9)
\]

Note that we neglect the covariance between two neighboring batches in eq. 5.9. This is justified because we take large periods and the sequence of costs constitutes a sequence of almost independent random variables.

Statistical inference

A paired-samples t-test is used to compare the differences in total cost of different order policies. The null hypothesis states that the difference \( \mu_d \) is zero \( (H_0 : \mu_d = 0 \text{ and } H_1 : \mu_d \neq 0) \) and the test statistic \( t \) has a \( t \)-distribution with \( n - 1 \) degrees of freedom.

\[
t = \frac{\bar{d}}{s_d \sqrt{\frac{1}{n}}} \quad (5.10)
\]

The order policy, given in table 5.5, will be compared with these of the standard EOQ, which orders each time 39 products, and the EOQ modified for inflation, which orders 37 units in a downward

\(^9\)Trajectory: a description of the evolution of the system over time.
5.5. STATISTICAL ANALYSIS OF RESULTS

market and 41 units in an upward market. This order policy is the white policy (Market 2) in figures 5.12 and 5.13.

Table 5.5: Order policy $Q^*$ market 2.

<table>
<thead>
<tr>
<th>Market</th>
<th>Bull market</th>
<th>Bear market</th>
</tr>
</thead>
<tbody>
<tr>
<td>Price level 1</td>
<td>65</td>
<td>52</td>
</tr>
<tr>
<td>Price level 2</td>
<td>61</td>
<td>40</td>
</tr>
<tr>
<td>Price level 3</td>
<td>58</td>
<td>33</td>
</tr>
<tr>
<td>Price level 4</td>
<td>54</td>
<td>30</td>
</tr>
<tr>
<td>Price level 5</td>
<td>50</td>
<td>29</td>
</tr>
<tr>
<td>Price level 6</td>
<td>46</td>
<td>29</td>
</tr>
<tr>
<td>Price level 7</td>
<td>43</td>
<td>29</td>
</tr>
<tr>
<td>Price level 8</td>
<td>39</td>
<td>29</td>
</tr>
<tr>
<td>Price level 9</td>
<td>35</td>
<td>28</td>
</tr>
<tr>
<td>Price level 10</td>
<td>32</td>
<td>27</td>
</tr>
</tbody>
</table>

Table 5.6 gives the parameters that are used in the simulation. The results are given in table 5.7.

Table 5.6: Overview of the simulation parameters.

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Period length $M$</td>
<td>5000 days</td>
</tr>
<tr>
<td>Warm up</td>
<td>1 period</td>
</tr>
<tr>
<td>Number of periods $n$</td>
<td>2000 batches</td>
</tr>
</tbody>
</table>

Table 5.7: Paired t-test results.

<table>
<thead>
<tr>
<th>Test</th>
<th>$\bar{d}$</th>
<th>$s_d$</th>
<th>$t - statistic$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$EOQ_{infl} - Q^*$</td>
<td>118.21</td>
<td>473.96</td>
<td>11.14</td>
</tr>
<tr>
<td>$EOQ - Q^*$</td>
<td>109.02</td>
<td>390.85</td>
<td>12.46</td>
</tr>
<tr>
<td>$EOQ - EOQ_{infl}$</td>
<td>39.72</td>
<td>349.35</td>
<td>5.08</td>
</tr>
</tbody>
</table>

The test indicates that all the null hypotheses can be rejected at the 1% level of significance. Thus the total cost $TC(Q^*)$ is significantly lower than $TC(EOQ)$ and $TC(EOQ_{infl})$. Finally, the assumptions of the paired-samples t-test need to be checked. The differences between pairs are assumed to be
normally distributed and independent of each other. The former is tested in SPSS. The Shapiro-Wilk test indicated that normality could not be rejected (p > 0.05). The independence is met due to the long period length.
Chapter 6

Conclusions and Future Research

6.1 Conclusion

In this master thesis some new inventory models that integrate non-stationary price evolution were created. The assumption of a constant price was dropped and substituted by different price evolutions. The goal was to develop and analyze the models for inventory management under these different market conditions. It was studied whether the classical policies were still optimal and what the total cost savings could be.

First, the literature was studied and extended with an investigation of the possible cost improvement under constant and deterministic exponential inflation. This could be seen as a study of the long-term behavior of the inventory system by using the long-term inflation rate. High demand items with a low fixed order cost / holding cost ratio turned out to have the smallest cost increase when ignoring inflation. For these products the inventory manager should not bother too much about inflation. This corresponds with the findings of Moon and Lee [20] where the effect of time-value of money and a random product life cycle is also included.

Second, a more stochastic price evolution was studied in the form of the geometric Brownian motion. The GBM is frequently invoked in studies as a model to capture the stochastic evolution of price levels. Its relevant properties and shortcomings are discussed. Monte-Carlo simulations were carried out and time-value of money was also considered. When accounting for the inflation rate, the same policy was found to be optimal as in the case of constant and deterministic exponential inflation. The volatility seemed to have no effect on the optimal policy but a lower total cost could be obtained under conditions of higher volatility. By accurately transforming the price fluctuations into a policy with
varying order quantities, the total cost decreases.

Because of the shortcomings of the geometric Brownian motion, a modified model was proposed which includes random price jumps and seasonality. It was found that under relatively high interest rates it is better to wait until the inventory is empty in order to keep the holding costs low and delay the order cost as much as possible. Even when it is known that the inflation rate will increase in the near future. In the opposite case, when the inflation rate is higher than the interest rate, well-timed orders can lead to cost reductions.

To make the model more realistic, stochastic demand and lead times were included. These variables were modeled as gamma distributions because it is only defined for non-negative values and the distribution is able to take multiple shapes depending on the shape factor. The optimal inventory system depends on multiple factors that all should be taken into account at the same time.

Since inventory management is a business field that already has been investigated by thousands of studies, the goal of this dissertation was not to find ground-breaking results but to give insights about the impact of different price evolution models. The investigation of an inventory system that is subject to a price evolution based on the GBM model did not lead to major results. The classical stock replenishment systems are quite robust against this stochastic price evolution. Even when it was assumed that the values of the inflation rates were known which is in practice not possible. It can also be concluded that the geometric Brownian motion is not really capable to capture all properties of the price evolution of a product. Constant expected return is not really realistic, prices are more likely to move to their medium-term average than move further away.

In chapter 5, short-term and mid-term price trends were studied. A bull and bear market respectively corresponded with an upward and a downward trend. The Markov decision process was stationary in the long-term but seemed non-stationary when a relatively short period was examined. A negative binomial demand distribution was considered to investigate the impact of variance on the order policies. Both analytical considerations and simulations were used to determine parameter values in such way that the price evolution model had a triangular wave pattern with a suitable transition rate.

The influence of the market condition, inflation rate, variance, etc. were investigated. Some general observations could been made. The optimal order quantity is for instance high at market bottoms and low at market tops, for both market conditions. In between the extrema, the order quantity is higher
than the EOQ for a bull market and lower than the EOQ for a bear market. For an upward trend it was found that the optimal order quantity slightly increases when the variance decreases. While the opposite is true for a downward trend. In case there is a capacity constraint the order quantity increases under some conditions because the system is not able to fully benefit of price drops.

Last, a simulation was carried out to compare the total cost of the optimal order policy with the standard EOQ policy adjusted for inflation. It was found that the cost reductions were rather small but significant.

### 6.2 Future research

The geometric Brownian motion may capture some but certainly not all the mechanisms of price evolution. Even with the modification for price jumps and seasonality \[17\] the model lacks to capture long range correlations and periods of constant price levels for instance. In \[13\] an alternative approach is proposed based on subordinated tempered stable geometric Brownian motion. Periods of stagnation can be observed with the inverse tempered stable subordinator responsible for trapping events. This alternative price model along with the relaxation of some of the assumptions can lead to a model that approaches the reality more. Furthermore, the model could also be extended with the impact of price discounts.

The case where one of the costs increases much faster than the other could also be investigated. In the dissertation, it was assumed that the costs move independent of each other over time, but with equal drift and variability. Thus in future research the unit purchase price may, for example, rise significantly faster than the average rate of inflation while the fixed order cost rises below average.

The MDP model that is studied in chapter 5 is subject to a lot of assumptions. A stochastic delivery lead time could be added to the model as well as the opportunity to order in states with a non-zero inventory level.
Bibliography


[22] Sivy, M., "If there’s no inflation, Why are prices up so much?", Time, Mar. 2013.


